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ТОЧНЫЕ РЕШЕНИЯ ДЛЯ КОНФОРМНО-ПЛОСКОЙ ВСЕЛЕННОЙ. II. ЛИНЕЙНОЕ УРАВНЕНИЕ СОСТОЯНИЯ И МНОГОМЕРНЫЕ МОДЕЛИ

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В рамках многомерного пространства-времени с одним временноподобным направлением рассмотрено получение конформно-плоских космологических моделей как точных решений уравнений тяготения для разных уравнений состояния с линейной связью между давлением и плотностью энергии. Утверждается, что такой подход приводит к выявлению некоторого дискретного набора уравнений состояния, для которых конформные множители тесно связаны с гармоническими функциями, являющимися решениями уравнений Лапласа в многомерных евклидовых пространствах целой размерности. Размерность этих пространств, в свою очередь, определяется конкретным уравнением состояния.

Ключевые слова: открытые космологические модели, точные решения гравитационных уравнений, многомерное пространство-время, линейное уравнение состояния, многомерное уравнение Лапласа.

EXACT SOLUTIONS OF THE CONFORMALLY FLAT UNIVERSE. II. THE LINEAR EQUATION OF STATE AND MULTIDIMENSIONAL MODELS

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The finding problem of conformally flat cosmological models as exact solutions of the equations of gravitation for different equations of state with linear connection between pressure and energy density is demonstrated within the limits of multidimensional space-time with one time-like direction. In this case the energy-momentum tensor (EMT) is taken as generalisation of EMT in an approach of the perfect Pascal fluid for space-time with four dimensions. The special case is EMT for an incoherent dust with zero pressure is related to the open model of Friedman's Universe.

It is claimed that such approach leads to an identification of some discrete set of equations of state for which conformal factors are connected with the harmonic functions as solutions of the Laplace equations in multidimensional Euclidean spaces of an integer dimensionality. Dimensionality of these spaces, in turn, is defined by a concrete equation of state. For four-dimensional space-times the corresponding table is constructed. This table allows to trace connection between a discrete set of linear equations of state and dimensionality of the auxiliary Euclidean spaces and also the functional expression of conformal factors of the open cosmological models related to potential functions, which are solutions of the Laplace equations in these auxiliary Euclidean spaces.

Thus it can be seen that three dimensional spatial-like manifold restricts a selection of discrete physically interpreted equations of state for the finding of exact solutions of the gravitation equations related to potential functions. Therefore, on the one hand, any linear equation of state can be approximated with any accuracy by any rational fraction. On the other hand, the exact solution of the many-dimensional equations of Einstein can be found only related via to potential functions when the spatial extension of space-time will be made up to necessary multidimension. Such possibility appears for any linear equation of state with a rational

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constant of proportionality at growth of the space dimensionality N $(N > 3)$. For such space-times the similar table is constructed, but without fixing of dimensionality of a spatial hypersurface. Here each value of spatial dimensionality N corresponds to $2N+1$ of linear equations of state. This table demonstrates the possibilities for each such equation of state with a rational constant of proportionality between pressure and density of energy under construction for any open cosmological model with the conformally flat metric, but in corresponding space-time with dimensionality more than four.

Keywords: the open cosmological models, exact solutions of the gravitation equations, multidimensional space-times, linear equation of state, equation of Laplace.

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Introduction

When studying conformally flat spaces based on the general theory of relativity, it is not uncommon to obtain and physically analyze the corresponding cosmological models in a synchronous reference frame, which is not always acceptable from the point of view of finding accurate analytical solutions to Einstein's equations due to a number of mathematical difficulties. For example, in such a frame of reference, it is impossible to obtain a final solution for an open cosmological model with incoherent dust and radiation. However, this problem is solved using a non-accompanying frame of reference ([1]- [3]), in which the 4-metric of space-time is conformally flat, and the conformal multiplier depends only on one variable that plays the role of distance in Minkowski 4-space (Fock's approach ([4], [5])). It turns out that such a description is equivalent to the introduction of a kinemetric frame of reference (see, for example, [6]- [8]) instead of a synchronous reference frame [9]. In addition, the use of a synchronous reference frame does not fully meet the symmetry of the cosmological problem, where the metric depends on a single variable (see, for example, [4]).

In the article [10] generalizing the Friedman model of the open universe [11], written in Fock form ([4], [5]), an exact analytical solution with an arbitrary state function is obtained, which is defined as the ratio of pressure to energy density. In particular cases, the state function coincides with the equations of state [10], including those representing a linear relationship between energy density and pressure.

In the previous article [1] a generalization of Friedman's solution for the open universe was found in the case of both matter and equilibrium light-like radiation (similar to electromagnetic radiation) with a pressure other than zero without introducing a specific equation of state. Here we will address directly the problem of the equation of state of matter in cosmological models. In our opinion, undeservedly little attention is paid to this problem, although the equation of state is an integral part of the Einstein system of equations in solving cosmological problems. Moreover, the successes of the inflationary approach in solving some problems of Friedman cosmology convince us that a wide variety of states of matter (both with positive and negative pressure) can be realized in the universe.

In our opinion, describing the evolution of the Universe in terms of changing the equation of state can be a very special approach that naturally combines the physical (thermodynamics, kinetics) and geometric (curvature, dimension of space-time) aspects of the theory of the evolution of the Universe. The success of the geometric multidimensional approach in combining fundamental interactions leads to the idea that the dimension of space-time should somehow manifest itself in physical quantities that characterize the evolution of the universe.

In this paper, within the framework of a multidimensional space-time with one time-like direction, we consider obtaining conformally flat models as exact solutions to Einstein's cosmological equations with nonzero pressure for different equations of state in the presence of a linear relationship between energy density and pressure. It turns out that this approach leads to the identification of a certain discrete set of equations of state for which conformal factors are closely related to harmonic functions, which are solutions of Laplace equations in multidimensional Euclidean spaces of integer dimension. The dimension of these spaces, in turn, is determined by a specific equation of state.

We will continue to use the Fock approach, in which the metric of 4-dimensional (and multidimensional) space-time is Galilean conformal, and the conformal multiplier is a function of one variable.

1. Multidimensional Einstein equations

We will try to answer this question within the framework of the Kaluza-Klein type approach, using a method that Y.S. Vladimirov [7] calls inductive. In other words, we will increase the number of space-like coordinates, leaving one time-like.

We will write the metric in the form similar to how it is done in [1]:

$$ds^2 = \exp(2\sigma)\,\delta_{\mu\nu}dx^\mu dx^\nu, \tag{1}$$

where $\exp(2\sigma)$ – the conformal multiplier; $\sigma = \sigma(S)$; $S^2 = \delta_{\mu\nu}x^\mu x^\nu$; $\delta_{\mu\nu} = diag(1; -1; -1; ...; -1)$ – the metric Minkowski tensor for multidimensional space-time; the speed of light and Newton's gravitational constant are equal to one, therefore, the Einstein gravitational constant here is $\varkappa = 8\pi$. But now the indexes are running through the values $\mu, \nu = 0, 1, \ldots, N$, where N – the number of spatial-like coordinates.

The equations of the gravitational field are postulated as a system of Einstein's equations (without the cosmological constant)

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\varkappa T_{\mu\nu} \tag{2}$$

where $G_{\mu\nu}$, $R_{\mu\nu}$ è R,– the Einstein, Ricci tensors and scalar curvature, respectively, constructed from a metric (1) in a similar way to the 4-dimensional case; the right-hand side is a multidimensional analogue of the energy-momentum tensor (TEI) of an ideal Pascal fluid, describing a multidimensional «liquid»)

$$T_{\mu\nu} = \varepsilon\, u_\mu u_\nu + p\, b_{\mu\nu}, \tag{3}$$

where ε – an analog of energy density; p – pressure analog; $(N+1)$-velocity $u_\mu = exp(\sigma) b_\mu$ proportional to the gradient of the variable S as a function of coordinates x^μ : $b_\mu = S_{,\mu}$; $u_\mu u^\mu = 1$ – the normalization condition $(N+1)$-velocity; $b_{\mu\nu} = u_\mu u_\nu - g_{\mu\nu}$ - N-subspace projector, which plays the role of a metric tensor for N-subspaces, In this case, the condition of orthogonality of the N-subspace and time-like congruence is fulfilled. u^μ : $b_{\mu\nu}u^\mu = 0$.

In fact, if $(N+1)$-velocity u^μ is defined as an analog of a monad in a multidimensional space of time by analogy with a 4-dimensional one (see, for example, [7]), then we can assume that TEI (3) is written according to the monadic approach.

On the other hand, the left side of the system (2) can be represented as

$$G_{\mu\nu} = (N-1)\left(\left[\sigma'' - \frac{\sigma'}{S} - (\sigma')^2\right] b_\mu b_\nu - \left[\sigma'' + \frac{N-1}{S}\sigma' + \frac{N-2}{2}(\sigma')^2\right]\delta_{\mu\nu}\right), \tag{4}$$

where the stroke denotes the derivative d/dS.

By performing the procedure $(1+N)$-splitting as it is done in 4 dimensions (see, for example, [7]), we obtain the following system of equations defining «energy density» and «pressure», and a special case of which, of course, are the equations ([1], (1.4)-(1.5))

$$N(N-1)\left[\frac{\sigma'}{S} + \frac{(\sigma')^2}{2}\right] = \varkappa\varepsilon\exp(2\sigma); \tag{5}$$

$$(N-1)\left[\sigma'' + \frac{N-1}{S}\sigma' + \frac{N-2}{2}(\sigma')^2\right] = -\varkappa p\, \exp(2\sigma). \tag{6}$$

We put the quotation marks because, strictly speaking, the meaning of these concepts in the case of a multidimensional space requires a completely separate consideration.

As is known, the open 4-dimensional Friedman model [11] is described by the equation of state of incoherent dust $p = 0$, which is a special case of the linear equation of state used in many cases in physics.

$$p(S) = \beta_0 \cdot \varepsilon(S), \tag{7}$$

where $|\beta_0| \leq 1$; $\beta_0 = const$.

A number of physically interpreted states of 4-dimensional matter are described by the relation $p \propto \varepsilon$: physical vacuum, incoherent dust, relativistic gas, etc., and correspond to specific discrete values of the parameter β_0.

The equation of state for multidimensional space-time, that is, the relationship between «pressure» and «energy density», is postulated in the simplest form (7), which, however, includes physically important cases.

In the future, we will not use quotation marks due to the fact that the meaning of the introduced concepts of energy density and pressure will be clear from the context.

The system of equations (5) and (6), taking into account (7) and excluding the energy density $\varepsilon(S)$, reduces to one equation

$$\sigma'' + [(N-1) + \beta_0 N]\,\frac{\sigma'}{S} + [(N-2) + \beta_0 N]\,\frac{(\sigma')^2}{2} = 0, \tag{8}$$

which, in turn, is a substitute for

$$\sigma(S) = \frac{2}{(N-2) + N\beta_0}\ln y(S) \tag{9}$$

for $\beta_0 \neq (2-N)/N$ and $\sigma(S) = y(S)$ for $\beta_0 = (2-N)/N$ easily converted to

$$y'' + \left(\frac{N(1+\beta_0) - 1}{S}\right)\, y' = 0. \tag{10}$$

Although this equation is trivially integrated, we will try to extract some different information from it. It is easy to see that in the case of an integer coefficient with the first derivative, the equation (10) is the radial part of the spherically symmetric Laplace equation in a multidimensional Euclidean space of dimension $k = N(1+\beta_0)$,

$$y'' + \frac{(k-1)}{S}y' = \frac{1}{S^{(k-1)}}\frac{\partial}{\partial S}\left(S^{(k-1)}\frac{\partial y}{\partial S}\right) = 0, \tag{11}$$

where the role of the radial variable is played by the same variable S, the value of which is preserved during transitions between different Euclidean spaces, just as in three-dimensional space the radius of a circle is the radius of a sphere built on this circle. The requirement of integers k is equivalent to the requirement of discreteness of values $\beta_0 = k/N - 1$, the number of which for each fixed N is equal to $(2N+1)$, and these values turn out to be rational (if the dimension of space-time is a natural number). In this case, the conformal multiplier is related to the function $y(S)$ as follows:

$$exp(2\sigma(S)) = y^{\frac{4}{k-2}}\ . \tag{12}$$

Thus, for integer values of k, the solutions of equation (11) turn out to be fundamental harmonic functions of the order of k (see, for example, [12])

$$y_k = \left(B_k + \frac{A_k}{S^{k-2}}\right), k \neq 2; \quad y_2 = \left(B_2 + A_2 \ln\left(\alpha S\right)\right), k = 2, \tag{13}$$

where the choice of constants is related to the specific physical asymptotics of the conformal multiplier. Moreover, strictly speaking, the k-dimensional spaces in which these functions will be harmonic are not necessarily Euclidean, since the satisfiability of the equation (11) with integers k can also be realized in a conformal-Euclidean space by requiring the scalar curvature of the latter to vanish. Indeed, if we consider a space of dimension n with a conformal-Euclidean metric [13]

$$g_{ab} = \exp\left(2\sigma\right)\ \delta_{ab} \tag{14}$$

where $a, b = 1, 2, \ldots, n;\ \delta_{ab} = (+1, +1, \ldots, +1)$, the scalar curvature of such a space is written as

$$R^{(n)} = 4\left(\frac{n-1}{n+1}\right)\left[\exp\left(-\frac{n+2}{2}\sigma\right)\right]\Delta_n y \tag{15}$$

where

$$\Delta_n y \equiv y'' + \frac{n-1}{S} y'$$

So, within the framework of our approach, we can state that in conformally flat models filled with matter with the equation of state (7), each type of substance (that is, each specific β_0 from a physically interpreted discrete set of values), determined by the values of N and k, corresponds to «its own» is a Euclidean (or conformal-Euclidean) space, the fundamental harmonic function of which determines the gravitational field (conformal multiplier) created by this substance. Thus, due to the uniqueness of the time-like direction, each type of substance has its own multidimensional space-time.

That such an interpretation makes sense can be seen by describing the conservation law of the energy-momentum tensor for our case. Having done this, we obtain the ratio

$$\varepsilon = \frac{const}{\left[S \exp(\sigma)\right]^n}, \tag{16}$$

where $n = N(1 + \beta_0)$ exactly matches the k entered earlier. The value $S \exp(\sigma(S)) = a(S)$ is nothing more than the radius of curvature of a space-like hypersurface in a synchronous reference frame. On the other hand (see, for example, [14])), the right-hand side can be interpreted as a value inversely proportional to the spatial volume of the system. This requires that n represents the dimension of the latter.

Here are two striking examples, in our opinion. For $N = 3$,
$beta_0 = 0$ (incoherent dust), we have $n = 3$, that is, the energy density is inversely proportional to the three-dimensional volume. For $N = 3$,
$beta_0 = 1/3$ (ultrarelativistic gas), we have $n = 4$, now the energy density is inversely proportional to the space-like four-dimensional volume. In other words, the state of matter determines the space characteristic of a given substance, in which the energy density decreases inversely with the volume. Moreover, it is easy to see that the only state for which the energy density will be finite on the cone $S = 0$ is the physical vacuum ($\beta_0 = -1$), regardless of from the dimension of N.

2. Four-dimensional space-time

Now we will specify the value of the dimension N. Since we are primarily interested in the manifold describing our universe, we assume $N = 3$ and consider which values of β_0 will be characteristic of this case (in the sense of our interpretation).

In this case, the dimension of the Laplacian, that is, the Euclidean (conformal-Euclidean) characteristic space, will be determined by the formula

$$n = 3(1 + \beta_0) \tag{17}$$

The discrete set of values β_0, which ensures the integers of the dimension (fundamental harmonic functions), will contain seven values starting from zero (recall that we limited ourselves to the condition $|\beta_0| \le 1$). In other words, a conformally flat 4-world with various β_0 associated with integers n corresponds to seven ($n = 0, 1, \ldots, 6$) Euclidean (conformal-Euclidean) spaces.

We write out for all values of n ($0 \le n \le 6$) conformal factors, pressure, four-dimensional and three-dimensional scalar curvatures, proper time as a function of the variable S, and a metric in a synchronous reference frame.

2.1. $n = 0,\ \beta_0 = -1$

The state of matter: physical vacuum (open de Sitter solution)

The conformal multiplier:

$$\exp(2\sigma) = \left(B + AS^2\right)^{-2}, \tag{18}$$

where A and B are constants. Obviously, you can put $B = 1$ everywhere, since this constant determines the choice of scale.

Knowing the conformal multiplier, it is easy to find pressure as a function of the variable S:

$$\varkappa p = 3\beta \left[\frac{2\sigma'}{S} + (\sigma')^2\right] \exp(-2\sigma). \tag{19}$$

After performing simple calculations, we get the expected result: the pressure is constant and depends only on constants.

$$\varkappa p = 12A, \tag{20}$$

from where it can be seen that the constant A is negative (the energy density should be positive).

The four-dimensional scalar curvature is found by the formula

$${}^4R = 6\exp(-2\sigma)\left[\sigma'' + \frac{3\sigma'}{S} + (\sigma')^2\right] = -48A > 0. \tag{21}$$

The three-dimensional scalar curvature in the case of the metric (1) in a non-corresponding frame of reference is calculated using a 3-projector $b_{ik} = u_i u_k - g_{ik}$ ($u^i = e^{-\sigma} b^i$ - 4-velocity; $b_i = S_{,i}$)and is equal to

$${}^3\hat{R} = -\frac{3\sigma'}{S}\exp(-2\sigma). \tag{22}$$

For a physical vacuum, this curvature takes the form

$${}^3\hat{R} = 6A\left(1 + AS^2\right). \tag{23}$$

At $S \to 0$
${}^3\hat{R}$ tends to the finite limit of ($6A$).

By converting $t = S\cosh R,\ r = S\sinh R$, the solution with the metric (1) is translated into the accompanying synchronous reference frame, where proper time is defined as

$$\tau = \int u_\mu dx^\mu = \int \exp[\sigma(S)]dS, \tag{24}$$

here u^μ - 4-velocity, which in the general case is not expressed in elementary functions (this, apparently, causes the reluctance of many to work with conformally flat metrics). The interval is written as

$$dS^2 = d\tau^2 - a^2(\tau)\left(dR^2 + \sinh^2 R d\Omega^2\right), \tag{25}$$

where $a(\tau) = e^\sigma S$ - the scale factor.

In our case, the proper time is easily found and is equal to

$$\tau = \frac{1}{\sqrt{|A|}} \ln\left|\frac{1+\sqrt{|A|}S}{1-\sqrt{|A|}S}\right| = \frac{1}{\sqrt{|A|}} \arctan h(\sqrt{|A|}S). \tag{26}$$

Choosing the area of change of τ from 0 to ∞, we get the area of change of $S:\ 0 \leq S \leq (1/\sqrt{|A|})$.

The corresponding scalar 3-curvature in the accompanying synchronous reference frame is obtained in the form

$$^3R = -\frac{1}{S^2}\left(1 - |A|S^2\right)^2 = -4|A| \sinh^2(2\sqrt{|A|}\tau), \tag{27}$$

and the interval is written in the standard form.

$$dS^2 = d\tau^2 - \left(\frac{1}{4\sqrt{|A|}}\right) \sinh^2(2\sqrt{|A|}\tau) d\ell^2. \tag{28}$$

2.2. $n = 1,\ \beta_0 = -2/3$

The state of matter: domain walls.

The conformal multiplier:

$$\exp(2\sigma) = (1 + AS)^{-4}. \tag{29}$$

Pressure:

$$\varkappa p = \frac{8A}{S}\left(1 + AS^2\right); \tag{30}$$

at the same time $A < 0$.

Four-dimensional scalar curvature:

$$^4R = -\left(\frac{36}{S}\right) A\left(1 + AS^2\right)^2. \tag{31}$$

Three-dimensional scalar curvature in an unrelated frame of reference:

$$^3\hat{R} = \left(\frac{6A}{S}\right)(1 + AS)^3. \tag{32}$$

Proper time:

$$\tau = \left(|A|\,(1 + AS)\right)^{-1}. \tag{33}$$

Metric element in a synchronous reference frame:

$$dS^2 = d\tau^2 - A^2\tau^4 d\ell^2. \tag{34}$$

2.3. $n = 2,\ \beta_0 = -1/3$

The state of matter: relativistic strings.
The conformal multiplier:

$$\exp(2\sigma) = S^{2A}. \tag{35}$$

Pressure:

$$\varkappa p = -A\,(A+2)\,S^{-2(A+1)}. \tag{36}$$

Four-dimensional scalar curvature:

$${}^4R = \frac{6A(A+2)}{S^{2(A+1)}}. \tag{37}$$

Three-dimensional scalar curvature in an unrelated frame of reference:

$${}^3\hat{R} = -\frac{3A}{S^{2(A+1)}}. \tag{38}$$

Proper time:

$$\tau = \frac{1}{(A+1)} S^{(A+1)}. \tag{39}$$

Metric element in a synchronous reference frame:

$$dS^2 = d\tau^2 - \big[(A+1)\tau\big]^{\left(\dfrac{2A}{A+1}\right)} d\ell^2. \tag{40}$$

It should be noted that this case is special both from the point of view of harmonic functions and from the point of view of the physical content of the resulting model: depending on the sign and magnitude of the constant A, both singular and non-singular models can be obtained.

2.4. $n = 3,\ \beta_0 = 0$

The state of matter: incoherent dust (Friedman's Universe).
The conformal multiplier:

$$\exp(2\sigma) = \left(1 + \frac{A}{S}\right)^4. \tag{41}$$

Pressure and energy density:

$$p = 0;\ \varkappa\varepsilon = -\frac{12A}{S^3\,(1+A/S)^6}. \tag{42}$$

which shows that the constant A is negative.

Four-dimensional scalar curvature:

$${}^4R = -\frac{12A}{S^3\,(1+A/S)^6}. \tag{43}$$

Three-dimensional scalar curvature in an unrelated frame of reference:

$${}^3\hat{R} = -\frac{6A}{S^3\,(1+A/S)^5}. \tag{44}$$

Proper time:

$$\tau = S - 2A\ln\left|\frac{S}{A}\right| - \frac{A^2}{S}. \qquad (45)$$

The scale factor in the synchronous reference system is not expressed directly in terms of proper time, as in the standard approach (see, for example, [14]).

2.5. $n = 4$, $\beta_0 = +1/3$

The state of matter: ultra-relativistic gas.

The conformal multiplier:

$$\exp(2\sigma) = \left(1 + A/S^2\right)^2. \qquad (46)$$

Pressure:

$$\varkappa p = -\frac{A}{S^4\left(1 + A/S^2\right)^4}. \qquad (47)$$

The four-dimensional scalar curvature ${}^4R = 0$, which obviously follows from the zero trace of the energy-momentum tensor in this case.

Three-dimensional scalar curvature in an unrelated frame of reference:

$${}^3\hat{R} = -\frac{6A}{S^4\left(1 + A/S^2\right)^3}. \qquad (48)$$

Proper time:

$$\tau = S + \frac{|A|}{S} - \left(\sqrt{|A|} + \frac{1}{\sqrt{|A|}}\right). \qquad (49)$$

The scale factor in the synchronous reference system:

$$dS^2 = d\tau^2 - \left(\tilde{\tau} - \frac{4|A|}{\tilde{\tau} - \frac{4|A|}{\sqrt{\tilde{\tau} - 4|A|}}}\right) d\ell^2;\ \tilde{\tau} = \tau + \left(\sqrt{A} + \frac{1}{\sqrt{A}}\right), \qquad (50)$$

from which it can be seen that the choice of the sign before the root and the values of the constant will be determined by the boundary conditions in a particular model, but for long times $a^2 \sim \tau$.

2.6. $n = 5$, $\beta_0 = +2/3$

The state of matter: non-relativistic degenerate gas.

The conformal multiplier:

$$\exp(2\sigma) = \left(1 + A/S^3\right)^{4/3}. \qquad (51)$$

Pressure:

$$\varkappa p = -\frac{8A}{S^5\left(1 + A/S^3\right)^{10/3}}; \qquad (52)$$

The four-dimensional scalar curvature:

$${}^4R = \frac{12A}{S^5\left(1 + A/S^3\right)^{10/3}}. \qquad (53)$$

Three-dimensional scalar curvature in an unrelated frame of reference:

$$ {}^3\hat{R} = \frac{6A}{S^5\big(1 + A/S^3\big)^{7/3}}. \tag{54} $$

Proper time as a function of S is not expressed in elementary functions.

2.7. $n = 6$, $\beta_0 = +1$

The state of matter: super-rigid state.
The conformal multiplier:

$$ \exp(2\sigma) = \big(1 + A/S^4\big). \tag{55} $$

Pressure:

$$ \varkappa p = -\frac{12A}{S^6\big(1 + A/S^4\big)^3}. \tag{56} $$

The four-dimensional scalar curvature:

$$ {}^4R = \frac{24A}{S^6\big(1 + A/S^4\big)^3}. \tag{57} $$

Three-dimensional scalar curvature in an unrelated frame of reference:

$$ {}^3\hat{R} = \frac{6A}{S^6\big(1 + A/S^4\big)^2}. \tag{58} $$

Table 1.

n	$n-1$	β_0	**The state of matter**	$\exp(2\sigma(\mathbf{S}))$
0	-1	-1	Physical vacuum	$(B + A\cdot S^2)^{-2}$
1	0	-2/3	Domain walls	$(B + A\cdot S)^{-4}$
2	1	-1/3	Relativistic strings	$A\cdot S^{2B}$
3	2	0	Incoherent dust	$(B + A/S)^4$
4	3	+1/3	Ultra-relativistic gas	$(B + A/S^2)^2$
5	4	+2/3	Non-relativistic degenerate gas	$(B + A/S^3)^{4/3}$
6	5	+1	Super-rigid state	$(B + A/S^4)$

So, in the case of a four-dimensional space-time, we have seven physically interpretable values of β_0 (equations of state of matter filling space-time) that allow us to describe the created gravitational fields (conformal multipliers of the 4-metric) in the language of fundamental harmonic functions of the corresponding Euclidean spaces. The main data obtained above for a four-dimensional space-time for comparison are summarized in Table 1.

3. Table of states of matter in multidimensional space-time

Now we turn again to the relation connecting β_0 from the equation of state, the dimension of the «reference» space-time $(N + 1)$ and the dimension of the fundamental harmonic function n. Since we need to have at least one time-like direction and one space-like direction in the «reference» variety (for the concept of spacetime to make sense), it is natural to start considering models with $N = 1$. Recall that for every fixed N, the number of associated Euclidean (conformal-Euclidean) spaces, that is, fundamental harmonic functions, is bounded and equal to $(2N + 1)$.

The full picture is clearly reflected in the form of Table 2, in which the values of N are laid out horizontally, and the dimension of the associated Euclidean (conformal-Euclidean) space n. Let us note some important points, in our opinion, characteristic of this table.

Table 2.

N/n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	0	-1/2	-2/3	-3/4	-4/5	-5/6	-6/7	-7/8	- 8/9	-9/10	-10/11
2	+1	0	-1/3	-2/4	-3/5	-4/6	-5/7	-6/8	-7/9	-8/10	-9/11
3		+1/2	0	-1/4	- 2/5	-3/6	-4/7	- 5/8	-6/9	-7/10	-8/11
4		+1	+1/3	0	-1/5	-2/6	-3/7	- 4/8	- 5/9	-6/10	-7/11
5			+2/3	+1/4	0	-1/6	-2/7	-3/8	-4/9	-5/10	-6/11
6			+1	+2/4	+1/5	0	-1/7	-2/8	-3/9	-4/10	-5/11
7				+3/4	+2/5	+1/6	0	-1/8	-2/9	-3/10	-4/11
8				+1	+3/5	+2/6	+1/7	0	-1/9	-2/10	-3/11
9					+4/5	+3/6	+2/7	1/8	0	-1/10	-2/11
10					+1	+4/6	+3/7	+2/8	+1/9	0	-1/11
11						+5/6	+4/7	+3/8	+2/9	+1/10	0

It is easy to see that only three «states» are present in all dimensions of space-time: «physical vacuum», «dust» and «super-rigid state». They are put in quotation marks due to the fact that the physical interpretation of specific values in different dimensions requires separate consideration. In this regard, it is significant that in the four-dimensional world (the third column), all the «possible» values of β have a clear physical interpretation and include all the basic states of matter used to find cosmological solutions.

It should be pointed out that harmonic functions of the same order (in other words, the same conformal multiplier) correspond to different values of β_0 in different dimensions. Then, in such a language, we can talk about some «unification» of the types of interactions, since for any β_0 (except for the physical vacuum, which is also very remarkable) there is a space-time in which the corresponding conformal multiplier will describe a model filled with a dust-type substance with $\beta_0 = 0$.

An interesting conclusion can be reached by referring again to 4-dimensional space-time: the transition to the 5th dimension «unifies» electromagnetic interaction (which, by the way, is quite consistent with the Kaluza-Klein approach); the transition to the 6th dimension «unifies» the state of matter with $\beta_0 = +2/3$, which is valid for all ideal systems: Boltzmann, bosonic, and fermionic (see, for example, [15]). That is, we are talking about electroweak interaction. The introduction of 7th dimension «turns to dust» a substance with a super-rigid equation of state in 4th matter, in other words, when a strong interaction is realized in 4th matter. This is consistent with the results of Y.S. Vladimirova [16], who showed that within the framework of the Kaluza-Klein approach, a 6-dimensional space-time is sufficient to combine the gravitational and electroweak interactions, and a 7-dimensional space-time with one time-like direction is sufficient to combine the gravitational and electroweak interactions.

On the other hand, the same value of β_0 can correspond to different orders of fundamental harmonic functions, that is , different conformal factors depending on the dimension of space-time. So $\beta_0 = +1/3$ is described by a fourth-order harmonic function in the 4th series, an eighth-order harmonic function in the 7th series, a twelfth-order harmonic function in the 10th series, and so on. Of course, one must remember that the specific value of β_0 corresponds to different physical conditions in different (in terms of dimension) worlds.

An essential point of this approach, in our opinion, is the possibility of describing models of a conformally flat universe filled with matter with an arbitrary β_0.

Conclusion

In this paper, within the framework of a multidimensional space-time with one time-like direction, we consider obtaining conformally flat cosmological models as exact solutions to the equations of gravity for various equations of state with a linear relationship between pressure and energy density. In this case, the energy-momentum tensor is taken as a generalization of TEI in the approximation of an ideal Pascal fluid in four dimensions. A special case of such a TEI is the TEI of incoherent dust with zero pressure, associated with the open Friedman model of the Universe. It turns out that the introduction of such an equation of state leads to the identification of a discrete physically interpretable set of equations of state for which conformal factors are closely related to harmonic functions, which are solutions of Laplace equations in multidimensional Euclidean spaces of integer dimension. The dimension of these spaces, in turn, is determined by a specific equation of state of matter.

For four-dimensional spacetimes, a corresponding table is provided that makes it possible to trace the relationship between a discrete set of linear equations of state, the dimension of the auxiliary Euclidean space, and the functional type of conformal multipliers of open cosmological models associated with harmonic functions satisfying the Laplace equations in these auxiliary Euclidean spaces. It can be seen that spatial three-dimensionality limits the choice of discrete physically interpretable equations of state to obtain accurate solutions to the equations of gravity associated with harmonic functions. Therefore, on the one hand, any linear equation of state can be approximated with any rational fraction with a predetermined accuracy, and, on the other hand, only by expanding space-time to the required dimension, one can write down the exact solution of the multidimensional Einstein equations by associating it with harmonic functions.

With an increase in the spatial dimension of N $(N > 3)$, this possibility appears for any linear equation of state with a rational coefficient of proportionality. A similar table is given for such spacetimes, but without fixing the dimension of the spatial hypersurface. In this case, each value of the spatial dimension N corresponds to $2N + 1$ linear equations of states. This table demonstrates the possibilities for each such equation of state with a rational coefficient of proportionality between pressure and energy density when constructing any open cosmological model with a conformally flat metric, but in the corresponding space-time with a dimension greater than four.

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