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ПРИМЕНЕНИЕ ТЕОРИИ ОБ ОБЩЕЙ α-НЕПОДВИЖНОЙ ТОЧКЕ В СЛУЧАЕ НЕЧЕТКОГО ОБОГАЩЕННОГО φ-ϕ-СЖИМАЮЩЕГО ОТОБРАЖЕНИЯ В $\mathcal{F}$-МЕТРИЧЕСКИХ ПРОСТРАНСТВАХ ДЛЯ РЕШЕНИЯ НЕЧЕТКИХ ИНТЕГРО-ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ

Ратхур Л.[a,1], Сингх В.[a,2], Раджи М.[b,3], Данджума Я. Д.[b,4], Ибрагим А.[b,5], Мишра Л. Н.[c,6], Мишра В. Н.[d,7]

[a] Кафедра математики, Национальный технологический институт, Чалтланг, Айджал 796 012, штат Мизорам, Индия.

[b] Кафедра математики, Научно-технический университет Конфлюэнс, Осара, штат Коги, Нигерия.

[c] Кафедра математики, Школа передовых наук, Технологический институт Веллуру, Веллуру, 632 014, штат Тамилнад, Индия.

[d] Кафедра математики, Национальный университет малых народов им. Индиры Ганди, Лалпур, Амаркантак, Ануппур, штат Мадхья-Прадеш 484 887, Индия.

Вводится понятие $\mathcal{F}$-метрического пространства. Рассматривается общая α-неподвижная точка для случая нечеткого обогащенного φ-ϕ сжимающего отображения в полном $\mathcal{F}$-метрическом пространстве. Исследование вносит вклад в теории нечетких множеств и неподвижных точек. Практическая применимость полученных теоретических результатов показана на наглядных примерах, а также на примере исследования возможности существования решения задачи о нечетких начальных значениях интегро-дифференциальных уравнений в контексте обобщенной производной, введенной Хукухарой.

Ключевые слова: общая α-неподвижная точка, $\mathcal{F}$-метрическое пространство, φ-ϕ сжимающее отображение, производная Хукухары.

COMMON α-FIXED POINT RESULTS FOR FUZZY ENRICHED φ-ϕ CONTRACTION IN $\mathcal{F}$-METRIC SPACES WITH APPLICATION TO FUZZY INTEGRO-DIFFERENTIAL EQUATIONS

Rathour L.[a,1], Singh V.[a,2], Raji M.[b,3], Danjuma Y. J.[b,4], Ibrahim A.[b,5], Mishra L. N.[c,6], Mishra V. N.[d,7]

[a] Department of Mathematics, National Institute of Technology, Chaltlang, Aizawl 796 012, Mizoram, India.

[b] Department of Mathematics, Confluence University of Science and Technology, Osara, Kogi State, Nigeria.

[c] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India.

[d] Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India.

[1]E-mail: laxmirathour817@gmail.com
[2]E-mail: vinaysingh14aug@gmail.com
[3]E-mail: rajimuhammed11@gmail.com
[4]E-mail: yahayajd@custech.edu.ng
[5]E-mail: Ibrahima@custech.edu.ng
[6]E-mail: lakshminarayanmishra04@gmail.com
[7]E-mail: vishnunarayanmishra@gmail.com

The paper aims to introduce the notion of $\mathcal{F}$-metric spaces and establish some common α-fixed point results for fuzzy enriched φ-ϕ contraction in a complete $\mathcal{F}$-metric spaces. These additions broaden the body of knowledge in the existing framework on fuzzy mappings and fixed point theory. Through illustrative examples, we demonstrate the practical applicability of our theoretical results. Also, we explore the existence of solution for fuzzy initial value problem of integro-differential equations in the context of generalized Hukuhara derivative, as an application.

Keywords: common α-fixed point, $\mathcal{F}$-metric space, φ-ϕ contraction mapping, Hukuhara derivative.

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Introduce

The Banach fixed point theorem [1], one of the foundational findings of fixed point theory, is important for determining the existence and uniqueness of solutions to many mathematical problems. Numerous problems encountered in everyday life stem from incomplete information that is not well-expressed in conventional mathematics. In 1965, Zadeh [1] introduced the notion of fuzzy sets, that proffer efficient ways to handle imprecise information, laying the foundation for subsequent research in fuzzy mathematics. Building upon Zadeh's work, Goguen [2] extended this concept of fuzzy set to L-fuzzy set thereby replacing the interval [0, 1] by L that is completely distributive lattice. Heilpern [3] further extended the notion of fuzzy mappings and derived fixed point results via metric linear space. For more, we refer [4–9].

Rashid et al. [10] introduced the conception of $\beta_{\mathcal{F}_L}$-admissible for two L-fuzzy mappings and derived numerous results for these mappings. Moreover, Jleli and Samet [11] expanded the classical metric space by introducing a modern metric space known as $\mathcal{F}$-metric space. Alnaser et al. [12] employed this idea in this direction and demonstrated a few fixed point results in $\mathcal{F}$-metric space with a few unresolved issues, including fixed point theorems for L-fuzzy mappings. The existence of fixed point results was established by Samet et al. [13] when a new class of contractive type mappings known as β-ψ contractive type mapping and β-admissible mappings in metric spaces was presented. Further, Raji [14] generalized the concept of β-ψ contractive type mappings and obtained various common fixed point results for this generalized class of contractive mappings. Further results can be found in [15–17].

Recently, Albargi [18] presented the idea of $\mathcal{F}$-metric space as an extension of traditional metric space, established some common α-fixed point results for rational β-ψ-contractions, and proved the Banach contraction principle in the context of this generalized metric space.

Based on the above insight, we introduce the concepts of $\mathcal{F}$-metric spaces and subsequently establish some common α-fixed point results for fuzzy enriched φ-ϕ contraction within the framework of complete $\mathcal{F}$-metric spaces. To bolster our findings, we offer illustrative examples demonstrating the practical application of the presented results. Also, we explored as an application, the existence of solution for fuzzy initial value problem of integro-differential equations in the context of generalized Hukuhara derivative.

1. Preliminaries

Definition 1.1 [19] A mapping $\phi : [0, \infty) \to [0, \infty)$ is said to be comparison function if the following are satisfied: (i) ϕ is monotonic increasing, (ii) $\lim\limits_{n\to\infty} \phi^n(t) = 0$, for each $t > 0$.

Definition 1.2 [12] A mapping $\phi : [0, \infty) \to [0, \infty)$ is said to be (c)-comparison function if it satisfies: (i) ϕ is monotonic increasing, (ii) for each $t > 0$, $\sum_{n=1}^{+\infty} \phi^n(t) < \infty$.

Remark 1.3 Let ϕ be a (c)-comparison function. Then (i) $\phi(0) = 0$, (ii) for each $t > 0$, $\phi(t) < t$, (iii) ϕ is right continuous at 0.

Definition 1.4 Let Φ be the collections of all function $\varphi : [0,\infty)^5 \to [0,\infty)$ that satisfies: (i) φ is continuous, (ii) $\varphi(t_1,t_2,t_3,t_4,t_5) = 0$ if and only if $t_1t_2t_3t_4t_5 = 0$.

Example 1.5 The following function $\varphi : [0,\infty)^5 \to [0,\infty)$ belong to Φ: (i) $\varphi(t_1,t_2,t_3,t_4,t_5) = t_1t_2t_3t_4t_5$, (ii) $\varphi(t_1,t_2,t_3,t_4,t_5) = e^{t_1t_2t_3t_4t_5} - 1$, (iii) $\varphi(t_1,t_2,t_3,t_4,t_5) = \ln(1 + t_1t_2t_3t_4t_5)$.

Definition 1.6 [20] A function T from a metric space (X,d) into itself is called ϕ-contraction if there exist a function $\phi : [0,\infty) \to [0,\infty)$ satisfying

$$d(Tx,Ty) \le \phi(d(x,y)),\ \forall x,\ y \in X. \tag{1}$$

Definition 1.7 A function T from a metric space (X,d) into itself is called (ϕ,φ)-contraction if there exist a function $\phi : [0,\infty) \to [0,\infty)$ and $\varphi : [0,\infty)^5 \to [0,\infty)$ satisfying

$$d(Tx,Ty) \le \phi(d(x,y)) - \varphi(d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)),\ \forall x,\ y \in X. \tag{2}$$

Definition 1.8 [15] A fuzzy set in X where X be a nonempty is a function $\Omega : X \to [0,1]$. If $x \in \Omega$, then $\Omega(x)$ is said to be the grade of membership of $x \in \Omega$. An α-level set of Ω denoted by $[\Omega]_\alpha$ define by

$$[\Omega]_\alpha = \{x : \Omega(x) \ge \alpha\},\ \text{if}\ \alpha \in (0,1],\ [\Omega]_0 = \overline{\{x : \Omega(x) > 0\}}. \tag{3}$$

If X is a metric space, then I^X is the collection of all fuzzy sets in X.

Definition 1.9 [21] An α-fuzzy fixed point of a fuzzy mapping $T : X \to I^X$ is defined as a point $x^* \in X$ where $\alpha \in (0,1]$ and $x^* \in [Tx^*]_{\alpha(x^*)}$.

Definition 1.10 [21] A common α-fuzzy fixed point of a fuzzy mappings $T, f : X \to I^X$ is defined as a point $x^* \in X$ where $\alpha \in (0,1]$ and $x^* \in [Tx^*]_{\alpha(x^*)} \bigcap [fx^*]_{\alpha(x^*)}$.

We now, introduce $\mathcal{F}$-metric space as follows: let $g : (0,+\infty) \to \mathbb{R}$ and $\mathcal{F}$ referred to the collections of functions g satisfying: $(\mathcal{F}_1)$ $\quad 0 < x < t \Rightarrow g(x) \le g(t)$, $(\mathcal{F}_2)$ for the sequence $\{x_n\} \subseteq \mathbb{R}^+$, $\lim\limits_{n\to\infty} x_n = 0 \Leftrightarrow \lim\limits_{n\to\infty} g(x_n) = -\infty$.

Definition 1.11 [11] Let $d_{\mathcal{F}} : X \times X \to [0,+\infty)$, where X is nonempty set. Suppose there exists $(g,h) \in \mathcal{F} \times [0,+\infty)$ such that (i) $(x,y) \in X \times X$, $d_{\mathcal{F}}(x,y) = 0 \Leftrightarrow x = y$, (ii) $d_{\mathcal{F}}(x,y) = d_{\mathcal{F}}(y,x)$, for all $(x,y) \in X \times X$, (iii) for every $(x,y) \in X \times X$, for every $N \in \mathbb{N}, N \ge 2$, and for each $(u_i)_{i=1}^N \subset X$ with $(u_1,u_N) = (x,y)$, we have

$$d_{\mathcal{F}}(x,y) > 0 \Rightarrow g(d_{\mathcal{F}}(x,y)) \le g[\Sigma_{i=1}^{N-1} d_{\mathcal{F}}(x_i,x_{i+1})] + h. \tag{4}$$

Then, $d_{\mathcal{F}}$ is referred to as $\mathcal{F}$-metric on X and $(X,d_{\mathcal{F}})$ called $\mathcal{F}$-metric space.

Example 1.12 [11] Let $d_{\mathcal{F}} : \mathbb{R} \times \mathbb{R} \to [0,+\infty)$ be a function define by

$$d_{\mathcal{F}}(x,y) = \begin{cases} (x-y)^2, & (x,y) \in [0,2] \times [0,2], \\ |x-y|, & (x,y) \notin [0,2] \times [0,2], \end{cases} \tag{5}$$

with $g(t) = \ln(t)$ and $h = \ln(2)$, is $\mathcal{F}$-metric.

Definition 1.13 [11] Let $(X,d_{\mathcal{F}})$ be $\mathcal{F}$-metric space. (i) Let $\{x_n\} \subseteq X$. If $\{x_n\}$ is convergent to x in $\mathcal{F}$-metric $d_{\mathcal{F}}$, then the sequence is $\mathcal{F}$-convergent to $x \in X$. (ii) The sequence $\{x_n\}$ is referred to as $\mathcal{F}$-Cauchy, if $\lim\limits_{n,m\to\infty} d_{\mathcal{F}}(x_n,x_m) = 0$. (iii) $(X,d_{\mathcal{F}})$ is $\mathcal{F}$-complete if every $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to $x \in X$.

Lemma 1.14 [12] Assume X_1 and X_2 be nonempty compact subsets of $\mathcal{F}$-metric space $(X,d_{\mathcal{F}})$ that is closed, if $x \in X_1$, then

$$d_{\mathcal{F}}(x,X_2) \le H_{\mathcal{F}}(X_1,X_2). \tag{6}$$

Definition 1.15 [9] Let $(X,d_{\mathcal{F}})$ be $\mathcal{F}$-metric space and $\beta : (X,d_{\mathcal{F}}) \times (X,d_{\mathcal{F}}) \to [0,+\infty)$. Let T, f be a pair of fuzzy mappings from X into $\mathcal{F}_L(X)$. Then, the pair (T,f) is referred to as an $\alpha_{\mathcal{F}}$-admissible if: (i) for a point $x \in X$ and $y \in [Tx]_{\alpha_T(x)}$, where $\alpha_T(x) \in (0,1]$ with $\beta(x,y) \ge 1$, then, $\beta(y,z) \ge 1$, for all $z \in [fy]_{\alpha_f(y)} \ne \emptyset$ where $\alpha_f(y) \in (0,1]$, (ii) for a point $x \in X$ and $y \in [fx]_{\alpha_T(x)}$, where $\alpha_f(x) \in (0,1]$ with $\beta(x,y) \ge 1$, then, $\beta(y,z) \ge 1$, for all $z \in [Ty]_{\alpha_T(y)} \ne \emptyset$ where $\alpha_T(y) \in (0,1]$.

2. Main results

Theorem 2.1. Let $(X, d_{\mathcal{F}})$ be $\mathcal{F}$-metric space and T, f be fuzzy mappings from X into I^X. Suppose for each $x \in X$, there exist $\alpha_T(x)$, $\alpha_f(x) \in (0,1]$ such that $[Tx]_{\alpha_T(x)}$, $[fx]_{\alpha_f(x)} \in C(2^X)$ satisfies: (i) $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete, (ii) for a point $x_0 \in X$ there exists $\alpha_T(x_0)$ or $\alpha_f(x_0) \in (0,1]$ such that $x_1 \in [Tx_0]_{\alpha_T(x_0)}$ or $x_1 \in [fx_0]_{\alpha_f(x_0)}$, (iii) for all $x, y \in X$, there exist $\phi \in \Psi$ and $\varphi \in \Phi$ such that

$$H_{\mathcal{F}}\big([Tx]_{\alpha_T(x)}, [fy]_{\alpha_f(y)}\big) \leq \phi(d_{\mathcal{F}}(x,y)) - \varphi\begin{pmatrix} d_{\mathcal{F}}(x,y), d_{\mathcal{F}}(x, [Tx]_{\alpha_T(x)}), d_{\mathcal{F}}(y, [fy]_{\alpha_f(y)}), \\ d_{\mathcal{F}}(x, [fy]_{\alpha_f(y)}), d_{\mathcal{F}}(y, [Tx]_{\alpha_T(x)}) \end{pmatrix}, \tag{7}$$

Then, T and f have a common α-fixed point $x^* \in [Tx^*]_{\alpha_T(x^*)} \cap [fx^*]_{\alpha_f(x^*)}$.

Proof. By condition (ii), we can choose a point $x_0 \in X$, there exists $\alpha_T(x_0) \in (0,1]$ such that $[Tx_0]_{\alpha_T(x_0)} \in C(2^X)$ is nonempty compact subset of X and there exists a point $x_1 \in [Tx_0]_{\alpha_T(x_0)}$ such that $d_{\mathcal{F}}(x_1, x_2) = d_{\mathcal{F}}(x_1, [Tx_0]_{\alpha_T(x_0)})$. Again, for x_1, there exists $\alpha_f(x_1) \in (0,1]$ such that $[fx_1]_{\alpha_f(x_1)} \in C(2^X)$. Since $[fx_1]_{\alpha_f(x_1)}$ is nonempty compact subset of X, there exists a point $x_2 \in [fx_1]_{\alpha_f(x_1)}$ such that $d_{\mathcal{F}}(x_1, x_2) = d_{\mathcal{F}}(x_1, [fx_1]_{\alpha_f(x_1)})$. With (7) and Lemma 1.14, we have

$$\begin{aligned} &d_{\mathcal{F}}(x_1, x_2) = d_{\mathcal{F}}(x_1, [fx_1]_{\alpha_f(x_1)}) \leq H_{\mathcal{F}}\big([Tx_0]_{\alpha_T(x_0)}, [fx_1]_{\alpha_f(x_1)}\big) \leq \\ &\phi(d_{\mathcal{F}}(x_0, x_1)) - \varphi\begin{pmatrix} d_{\mathcal{F}}(x_0, x_1), d_{\mathcal{F}}(x_0, [Tx_0]_{\alpha_T(x_0)}), d_{\mathcal{F}}(x_1, [fx_1]_{\alpha_f(x_1)}), \\ d_{\mathcal{F}}(x_0, [fx_1]_{\alpha_f(x_1)}), d_{\mathcal{F}}(x_1, [Tx_0]_{\alpha_T(x_0)}) \end{pmatrix} \leq \\ &\phi(d_{\mathcal{F}}(x_0, x_1)) - \varphi(d_{\mathcal{F}}(x_0, x_1), d_{\mathcal{F}}(x_0, x_1), d_{\mathcal{F}}(x_1, x_2), d_{\mathcal{F}}(x_0, x_2), d_{\mathcal{F}}(x_1, x_1)). \end{aligned} \tag{8}$$

Apply definition 1.4 in (8), we have

$$d_{\mathcal{F}}(x_1, x_2) \leq \phi(d_{\mathcal{F}}(x_0, x_1)). \tag{9}$$

Now, for $x_2 \in X$, there exists $\alpha_T(x_2) \in (0,1]$ such that $[Tx_2]_{\alpha_T(x_2)} \in C(2^X)$. Since $[Tx_2]_{\alpha_T(x_2)}$ is nonempty compact subset of X, there exists a point $x_3 \in [Tx_2]_{\alpha_T(x_2)}$ such that $d_{\mathcal{F}}(x_2, x_3) = d_{\mathcal{F}}(x_2, \alpha_T(x_2))$. Again by (7) and Lemma 1.14, we have

$$\begin{aligned} &d_{\mathcal{F}}(x_2, x_3) = d_{\mathcal{F}}(x_2, [Tx_2]_{\alpha_T(x_2)}) \leq H_{\mathcal{F}}\big([fx_1]_{\alpha_f(x_1)}, [Tx_2]_{\alpha_T(x_2)}\big) \leq \\ &\phi(d_{\mathcal{F}}(x_2, x_1)) - \varphi\begin{pmatrix} d_{\mathcal{F}}(x_2, x_1), d_{\mathcal{F}}(x_2, [Tx_2]_{\alpha_T(x_2)}), d_{\mathcal{F}}(x_1, [fx_1]_{\alpha_f(x_1)}), \\ d_{\mathcal{F}}(x_2, [fx_1]_{\alpha_f(x_1)}), d_{\mathcal{F}}(x_1, [Tx_2]_{\alpha_T(x_2)}) \end{pmatrix} \leq \\ &\phi(d_{\mathcal{F}}(x_2, x_1)) - \varphi(d_{\mathcal{F}}(x_2, x_1), d_{\mathcal{F}}(x_2, x_3), d_{\mathcal{F}}(x_1, x_2), d_{\mathcal{F}}(x_2, x_2), d_{\mathcal{F}}(x_1, x_3)). \end{aligned} \tag{10}$$

Again, apply definition 1.4 in (10), we have

$$d_{\mathcal{F}}(x_2, x_3) \leq \phi(d_{\mathcal{F}}(x_1, x_2)). \tag{11}$$

Continuing this process having chosen $x_1, x_2, x_3, x_4 \ldots$, we establish a sequence $\{x_n\}$ in X such that $x_{2n+1} \in [Tx_{2n}]_{\alpha_T(x_{2n})}, x_{2n+2} \in [fx_{2n+1}]_{\alpha_f(x_{2n+1})}$, then for all n, we have

$$d_{\mathcal{F}}(x_{2n+1}, x_{2n+2}) \leq \phi(d_{\mathcal{F}}(x_{2n}, x_{2n+1})), \tag{12}$$

and

$$d_{\mathcal{F}}(x_{2n+2}, x_{2n+3}) \leq \phi(d_{\mathcal{F}}(x_{2n+1}, x_{2n+2})). \tag{13}$$

With (12) and (13), we obtain

$$d_{\mathcal{F}}(x_n, x_{n+1}) \leq \phi(d_{\mathcal{F}}(x_{n-1}, x_n)) \leq \ldots \leq \phi^n(d_{\mathcal{F}}(x_0, x_1)). \tag{14}$$

Let $\varepsilon > 0$ and $n(\varepsilon) \in \mathbb{N}$ and $(g,h) \in \mathcal{F} \times [0, +\infty)$ be such that (iii) of definition 1.11 is satisfied. Again, let $\varepsilon > 0$ be fixed. By $\mathcal{F}_1$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow g(t) < g(\delta) - h. \tag{15}$$

Let $n(\varepsilon) \in \mathbb{N}$ such that $\sigma_{n\geq n(\varepsilon)}\phi^n(d_{\mathcal{F}}(x_0,x_1)) < \delta$. By (15) and $\mathcal{F}_1$, we have

$$g\bigg(\sum_{i=n}^{m-1}\phi^n(d_{\mathcal{F}}(x_0,x_1))\bigg) \leq g\bigg(\sum_{n\geq n(\varepsilon)}\phi^n(d_{\mathcal{F}}(x_0,x_1))\bigg) < g(\varepsilon) - h. \tag{16}$$

We have for all $m > n \geq \mathbb{N}$,

$$g(d_{\mathcal{F}}(x_0,x_1)) \leq g\bigg(\sum_{i=n}^{m-1} d_{\mathcal{F}}(x_i,x_{i+1})\bigg) + h \leq g\bigg(\sum_{i=n}^{m-1}\phi^n(d_{\mathcal{F}}(x_i,x_{i+1}))\bigg) + h \leq$$
$$g\bigg(\sum_{n\geq n(\varepsilon)}\phi^n(d_{\mathcal{F}}(x_i,x_{i+1}))\bigg) + h \leq g(\varepsilon). \tag{17}$$

By $\mathcal{F}_1$, we have $d_{\mathcal{F}}(x_n,x_m) < \varepsilon, m > n \geq \mathbb{N}$. It follow that the sequence $\{x_n\}$ is $\mathcal{F}$-Cauchy. Since $(X,d_{\mathcal{F}})$ is $\mathcal{F}$-complete, there exists $x^* \in X$ such that the sequence $\{x_n\}$ is $\mathcal{F}$-convergence to x^*, that is,

$$\lim_{n\to\infty} d_{\mathcal{F}}(x_n,x^*) = 0. \tag{18}$$

To show that $x^* \in [Tx^*]_{\alpha_T(x^*)}$, we let $d_{\mathcal{F}}(x^*,[Tx^*]_{\alpha_T(x^*)}) > 0$. By definition of g and (iii) of definition 1.11, we have

$$g\big(d_{\mathcal{F}}(x^*,[Tx^*]_{\alpha_T(x^*)})\big) \leq g\big(d_{\mathcal{F}}(x^*,x_{2n}) + d_{\mathcal{F}}\big(x_{2n},[Tx^*]_{\alpha_T(x^*)}\big)\big) + h \leq$$
$$g\big(d_{\mathcal{F}}(x^*,x_{2n}) + H_{\mathcal{F}}\big([fx_{2n-1}]_{\alpha_f(x_{2n-1})},[Tx^*]_{\alpha_T(x^*)}\big)\big) + h \leq g\Bigg(d_{\mathcal{F}}(x^*,x_{2n}) + \phi(d_{\mathcal{F}}(x^*,x_{2n-1})) -$$
$$\varphi\begin{pmatrix} d_{\mathcal{F}}(x^*,x_{2n-1}), d_{\mathcal{F}}(x^*,[Tx^*]_{\alpha_T(x^*)}), d_{\mathcal{F}}(x_{2n-1},[fx_{2n-1}]_{\alpha_f(x_{2n-1})}), \\ d_{\mathcal{F}}(x^*,[fx_{2n-1}]_{\alpha_f(x_{2n-1})}), d_{\mathcal{F}}(x_{2n-1},[Tx^*]_{\alpha_T(x^*)}) \end{pmatrix}\Bigg) + h \leq g\Bigg(d_{\mathcal{F}}(x^*,x_{2n}) +$$
$$\phi(d_{\mathcal{F}}(x^*,x_{2n-1})) - \begin{pmatrix} d_{\mathcal{F}}(x^*,x_{2n-1}), d_{\mathcal{F}}(x^*,[Tx^*]_{\alpha_T(x^*)}), d_{\mathcal{F}}(x_{2n-1},x_{2n}), \\ d_{\mathcal{F}}(x^*,x_{2n}), d_{\mathcal{F}}(x_{2n-1},[Tx^*]_{\alpha_T(x^*)}) \end{pmatrix}\Bigg) + h \tag{19}$$

Since $\{x_n\}$ is $\mathcal{F}$-convergent to x^*, so by the properties of $\phi \in \Psi$, $\varphi \in \Phi$, (18) with $\mathcal{F}_2$ and on taking limit in (19), we have

$$\lim_{n\to\infty} g\big(d_{\mathcal{F}}(x^*,[Tx^*]_{\alpha_T(x^*)})\big) = \lim_{n\to\infty} g\big(d_{\mathcal{F}}(x^*,x_{2n}) + d_{\mathcal{F}}(x^*,x_{2n-1}) + h\big) = -\infty, \tag{20}$$

a contradiction. Therefore, we get $d_{\mathcal{F}}(x^*,[Tx^*]_{\alpha_T(x^*)}) = 0$, which implies $x^* \in [Tx^*]_{\alpha_T(x^*)}$.

Similarly, we can show that $x^* \in [fx^*]_{\alpha_f(x^*)}$, we let $d_{\mathcal{F}}(x^*,[fx^*]_{\alpha_f(x^*)}) > 0$. By definition of g and (iii) of definition 1.11, we have

$$g\big(d_{\mathcal{F}}(x^*,[fx^*]_{\alpha_f(x^*)})\big) \leq g\big(d_{\mathcal{F}}(x^*,x_{2n+1}) + d_{\mathcal{F}}\big(x_{2n+1},[fx^*]_{\alpha_f(x^*)}\big)\big) + h \leq$$
$$g\big(d_{\mathcal{F}}(x^*,x_{2n+1}) + H_{\mathcal{F}}\big([Tx_{2n}]_{\alpha_T(x_{2n})},[fx^*]_{\alpha_f(x^*)}\big)\big) + h \leq g\Bigg(d_{\mathcal{F}}(x^*,x_{2n+1}) + \phi(d_{\mathcal{F}}(x_{2n},x^*)) -$$
$$\varphi\begin{pmatrix} d_{\mathcal{F}}(x_{2n},x^*), d_{\mathcal{F}}(x_{2n},[Tx_{2n}]_{\alpha_T(x_{2n})}), d_{\mathcal{F}}(x^*,[fx^*]_{\alpha_f(x^*)}), \\ d_{\mathcal{F}}(x_{2n},[fx^*]_{\alpha_f(x^*)}), d_{\mathcal{F}}(x^*,[Tx_{2n}]_{\alpha_T(x_{2n})}) \end{pmatrix}\Bigg) + h \leq g\Bigg(d_{\mathcal{F}}(x^*,x_{2n+1}) +$$
$$\phi(d_{\mathcal{F}}(x_{2n},x^*)) - \begin{pmatrix} d_{\mathcal{F}}(x_{2n},x^*), d_{\mathcal{F}}(x_{2n},x_{2n+1}), d_{\mathcal{F}}(x^*,[fx^*]_{\alpha_f(x^*)}), \\ d_{\mathcal{F}}(x_{2n},[fx^*]_{\alpha_f(x^*)}), d_{\mathcal{F}}(x^*,x_{2n+1}) \end{pmatrix}\Bigg) + h \tag{21}$$

Since $\{x_n\}$ is $\mathcal{F}$-convergent to x^*, so by the properties of $\phi \in \Psi$, $\varphi \in \Phi$, (18) with $\mathcal{F}_2$ and on taking limit in (21), we have

$$\lim_{n\to\infty} g\big(d_{\mathcal{F}}(x^*,[fx^*]_{\alpha_f(x^*)})\big) = \lim_{n\to\infty} g\big(d_{\mathcal{F}}(x^*,x_{2n}) + d_{\mathcal{F}}(x^*,x_{2n-1}) + h\big) = -\infty, \tag{22}$$

a contradiction. Hence, $d_{\mathcal{F}}(x^*,[Tf]_{\alpha_f(x^*)}) = 0$, which implies $x^* \in [fx^*]_{\alpha_f(x^*)}$.

Combining the results from (19) and (21), we have

$$x^* \in [Tx^*]_{\alpha_T(x^*)} \cap [fx^*]_{\alpha_f(x^*)}. \tag{23}$$

Thus, T and f have a common α-fixed point $x^* \in [Tx^*]_{\alpha_T(x^*)} \cap [fx^*]_{\alpha_f(x^*)}$.

Example 2.2. Consider $X = [0, +\infty)$, for all $x,y \in X$, $\mathcal{F}$-metric $d_{\mathcal{F}} : X \times X \to R_0^+$ is define

$$d_{\mathcal{F}}(x,y) = \begin{cases} (x-y)^2, & (x,y) \in [0,4] \times [0,4], \\ |x-y|, & (x,y) \notin [0,4] \times [0,4], \end{cases} \tag{24}$$

and for $t > 0$ and $h = \ln(4)$,$g(t) = \ln(t)$. Then, $(X, d_{\mathcal{F}})$ admits $\mathcal{F}$-complete $\mathcal{F}$-metric space and not a metric space because $d_{\mathcal{F}}$ does not fulfill the triangular inequality as

$$d_{\mathcal{F}}(1,4) > 5 = d_{\mathcal{F}}(1,3) + d_{\mathcal{F}}(3,4). \tag{25}$$

Moreover, let $\alpha(x) \in (0,1]$ and $T, f : X \to I^X$ as (i) If $x = 0$,

$$T(x)(t) = \begin{cases} 1, & t \in 0, \\ 0, & t \notin 0, \end{cases} \tag{26}$$

(ii) If $0 < x < \infty$,

$$T(x)(t) = \begin{cases} \alpha, & 0 \le t < \frac{x^2}{60}, \\ \frac{\alpha}{3}, & \frac{x^2}{60} \le t < \frac{x^2}{30}, \\ \frac{\alpha}{6}, & \frac{x^2}{30} \le t < x^2, \\ 0, & x^2 \le t < \infty. \end{cases}, \quad T(x)(t) = \begin{cases} \alpha, & 0 \le t < \frac{y^2}{40}, \\ \frac{\alpha}{3}, & \frac{y^2}{40} \le t < \frac{y^2}{30}, \\ \frac{\alpha}{24}, & \frac{y^2}{30} \le t < y^2, \\ 0, & y^2 \le t < \infty. \end{cases}. \tag{27}$$

We now define $\phi : [0,\infty) \to [0,\infty)$ by $\phi(t) = t/2$, for $t > 0$ with $\phi \in \Psi$ and $\varphi : [0,\infty)^5 \to [0,\infty)$ by $\phi(t) = t/3$ with $\varphi \in \Phi$.

For all $x \in X$, there exists $\alpha_T(x) = (\alpha/3) \in (0,1]$ and $\alpha_f(x) = (\alpha/6) \in (0,1]$, such that $[Tx]_{\alpha_T(x)}, [fx]_{\alpha_f(x)} \in C(2^X)$.

Case I: If $x = y = 0$, then $[Tx]_{(\alpha/3)} = [fy]_{(\alpha/6)} = 0$, then

$$H_{\mathcal{F}}\Big([Tx]_{\alpha_T(x)}, [fy]_{\alpha_f(x)}\Big) = 0 \le \phi(d_{\mathcal{F}}(x,y)) - \varphi\begin{pmatrix} d_{\mathcal{F}}(x,y), d_{\mathcal{F}}(x,[Tx]_{\alpha_T(x)}), d_{\mathcal{F}}(y,[fy]_{\alpha_f(y)}), \\ d_{\mathcal{F}}(x,[fy]_{\alpha_f(y)}), d_{\mathcal{F}}(y,[Tx]_{\alpha_T(x)}) \end{pmatrix}. \tag{28}$$

Case II: $x, y \in (0,\infty)$, then

$$[Tx]_{\alpha/3} = \left\{t \in X : T(x)(t) \ge \frac{\alpha}{3}\right\} = \left[0, \frac{x^2}{6}\right] \text{ and } [Ty]_{\alpha/3} = \left\{t \in X : T(y)(t) \ge \frac{\alpha}{3}\right\} = \left[0, \frac{y^2}{6}\right]. \tag{29}$$

For $x \neq y$ and by definition of $d_{\mathcal{F}}$, we have

$$\begin{aligned} H_{\mathcal{F}}([Tx]_{\alpha_T(x)}, [fy]_{\alpha_f(y)}) = \left(\frac{x^2}{30} - \frac{y^2}{30}\right)^2 \le \left|\frac{x+y}{30}(x-y)\right|^2 \le \frac{1}{12}|x-y|^2 = \frac{1}{12} d_{\mathcal{F}}(x,y) \le \\ \phi(d_{\mathcal{F}}(x,y)) - \varphi\begin{pmatrix} d_{\mathcal{F}}(x,y), d_{\mathcal{F}}(x,[Tx]_{\alpha_T(x)}), d_{\mathcal{F}}(y,[fy]_{\alpha_f(y)}), \\ d_{\mathcal{F}}(x,[fy]_{\alpha_f(y)}), d_{\mathcal{F}}(y,[Tx]_{\alpha_T(x)}) \end{pmatrix}. \end{aligned} \tag{30}$$

Hence, all the conditions of Theorem 2.1 are satisfied. Thus, there exists $0 \in [0,+\infty)$, that is, $0 \in [T0]_{(\alpha/3)} \cap [f0]_{(\alpha/6)}$.

Corollary 2.3 Let $(X, d_{\mathcal{F}})$ be $\mathcal{F}$-metric space and T be fuzzy mappings from X into I^X. Suppose for each $x \in X$, there exist $\alpha_T(x) \in (0,1]$ such that $[Tx]_{\alpha_T(x)} \in C(2^X)$ satisfies: (i) $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete, (ii) for a point $x_0 \in X$ there exists $\alpha_T(x_0) \in (0,1]$ such that $x_1 \in [Tx_0]_{\alpha_T(x_0)}$, (iii) for all $x, y \in X$, there exist $\phi \in \Psi$ and $\varphi \in \Phi$ such that

$$H_{\mathcal{F}}([Tx]_{\alpha_T(x)}, [Ty]_{\alpha_f(y)}) \le \phi(d_{\mathcal{F}}(x,y)) - \varphi\begin{pmatrix} d_{\mathcal{F}}(x,y), d_{\mathcal{F}}(x,[Tx]_{\alpha_T(x)}), d_{\mathcal{F}}(y,[Ty]_{\alpha_T(y)}), \\ d_{\mathcal{F}}(x,[Ty]_{\alpha_T(y)}), d_{\mathcal{F}}(y,[Tx]_{\alpha_T(x)}) \end{pmatrix}. \tag{31}$$

Then, T has a fixed point $x^* \in [Tx^*]_{\alpha(x^*)}$.

Proof The proof follows from Theorem 2.1 by taking one fuzzy mapping from X into I^X.

We now consider the fixed points results for multivalued mappings.

Corollary 2.4 Let $(X, d_{\mathcal{F}})$ be $\mathcal{F}$-metric space Let R be a fuzzy mapping from $(X, d_{\mathcal{F}})$ into $CB(X)$ satisfies: (i) $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete, (ii) for a point $x_0 \in X$ there exists $x_1 \in Rx_0$, (iii) for all $x, y \in X$, there exist $\phi \in \Psi$ and $\varphi \in \Phi$ such that

$$H_{\mathcal{F}}(Rx, Ry) \leq \phi(d_{\mathcal{F}}(x, y)) - \varphi(d_{\mathcal{F}}(x, y), d_{\mathcal{F}}(x, Rx), d_{\mathcal{F}}(y, Ry), d_{\mathcal{F}}(x, Ry), d_{\mathcal{F}}(y, Rx)), \tag{32}$$

Then, R has a fixed point $x^* \in Rx^*$.

Proof Define α-fuzzy mappings $T : X \to I^X$, for some $\alpha_T \in (0, 1]$ by

$$T(x)(t) = \begin{cases} \alpha_T, & t \in R_1 x, \\ 0, & t \notin R_1 x. \end{cases} \tag{33}$$

Then,

$$[Tx]_{\alpha_T(x)} = \{t \in X : T(x)(t) \geq \alpha_T(x)\} = R_1 x. \tag{34}$$

Implies for all $x, y \in X$,

$$H_{\mathcal{F}}([Tx]_{\alpha_T(x)}, [Ty]_{\alpha_T(y)}) = H_{\mathcal{F}}(R_1 x, R_2 y). \tag{35}$$

The proof the rest follows from Corollary 2.3. Thus $x^* \in X$, $x^* \in [Tx^*]_{\alpha_T(x^*)} = Rx^*$.

Corollary 2.5 Let $(X, d_{\mathcal{F}})$ be $\mathcal{F}$-metric space and R_1, R_2 be fuzzy mappings from $(X, d_{\mathcal{F}})$ into $CB(X)$ satisfies: (i) $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete, (ii) for a point $x_0 \in X$, there exists $x_1 \in R_1 x_0$ or $x_1 \in R_2 x_0$, (iii) for all $x, y \in X$, there exist $\phi \in \Psi$ and $\varphi \in \Phi$ such that

$$H_{\mathcal{F}}(R_1 x, R_2 y) \leq \psi(d_{\mathcal{F}}(x, y)) - \varphi\left(d_{\mathcal{F}}(x, y), d_{\mathcal{F}}(x, R_1 x), d_{\mathcal{F}}(y, R_2 y), d_{\mathcal{F}}(x, R_2 y), d_{\mathcal{F}}(y, R_1 x)\right). \tag{36}$$

Then, R_1 and R_2 have a common fixed point $x^* \in R_1 x^* \cap R_2 x^*$.

Proof Define α-fuzzy mappings $T, f : X \to I^X$, for some $\alpha_T, \alpha_f \in (0, 1]$ by

$$T(x)(t) = \begin{cases} \alpha_T, & t \in R_1 x, \\ 0, & t \notin R_1 x. \end{cases} \text{and } f(y)(t) = \begin{cases} \alpha_f, & t \in R_2 y, \\ 0, & t \notin R_2 y. \end{cases} \tag{37}$$

Then,

$$[Tx]_{\alpha_T(x)} = \{t \in X : T(x)(t) \geq \alpha_T(x)\} = R_1 x, \tag{38}$$

and

$$[fx]_{\alpha_f(x)} = \{t \in X : f(x)(t) \geq \alpha_f(x)\} = R_2 y. \tag{39}$$

Implies for all $x, y \in X$,

$$H_{\mathcal{F}}([Tx]_{\alpha_T(x)}, [fy]_{\alpha_f(y)}) = H_{\mathcal{F}}(R_1 x, R_2 y). \tag{40}$$

The proof of the rest follows from Theorem 2.1. Thus $x^* \in X$, $x^* \in [Tx^*]_{\alpha_T(x^*)} \cap [fx^*]_{\alpha_f(x^*)} = R_1 x^* \cap R_2 x^*$.

3. Application

Here, we demonstrate applicability of the results developed in the previous sections to investigate the existence of solution with fuzzy initial value problem via generalized Hukuhara differentiability. For more on this, we refer to [22–25].

As a starting point, we introduce the symbols that will be use in this section.

Let $\mathcal{H}_c^n$ be denote the space of compact, nonempty and convex subsets of the n-dimensional Euclidean space $\mathbb{R}^n$. If $A, B \in \mathcal{H}_c^n$ and $||.||$ denotes Euclidean norm in $\mathbb{R}^n$, then, the Hausdorff metric d on $\mathcal{H}_c^n$ is define as

$$d(A, B) = max \left\{ \sup_{a \in A} \inf_{b \in B} ||a - b||, \sup_{b \in B} \inf_{a \in A} ||a - b|| \right\}, \tag{41}$$

and we introduce the following definitions.

Definition 3.1 Let $u : \mathbb{R}^n \to [0, 1]$ be a fuzzy mapping. (a) u is said to be normal, if there exists $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$. (b) u is called to be fuzzy convex, if for all $x, y \in \mathbb{R}^n$ and $0 \le \mu \le 1$, we have

$$u(\mu x + (1 - \mu)y) \ge min\{u(x), u(y)\}, \tag{42}$$

(c) u is said to be upper semicontinuous, if for all $\alpha \in [0, 1]$, $[u]^\alpha$ is closed. (d) $[u]^0$ is compact.

Definition 3.2 [25] Suppose $u, v, w \in \mathcal{F}^n$. An element w is referred to as Hukuhara difference of u and v, if it satisfies the equation $u = v + w$. Now, $u \ominus_H v$ denotes the Hukuhara difference points of u and v. Clearly, $u \ominus_H u = \{0\}$, and if $u \ominus_H v$ exists, then this unique.

Lemma 3.3 [24] Let $T, f : [a, b] \to C^1$, where C^1 is the collections of fuzzy number in R and $\eta \in R$, the

$$\begin{gathered}
\text{(i) } \int_a^b (T + f)(t)dt = \int_a^b T(t)dt + \int_a^b f(t)dt, \\
\text{(ii) } \int_a^b \eta T(t)dt = \eta \int_a^b T(t)dt, \\
\text{(iii) } d_\infty(T(t), f(t)) \text{ is integrable}, \\
\text{(iv)} d_\infty(\int_a^b T(t)dt, \int_a^b f(t)dt) \le \int_a^b d_\infty(T(t)dt, f(t)dt), \text{ for } t \in [a, b].
\end{gathered}$$

Definition 3.4 [25] Assume $g : (a, b) \to \mathcal{F}^n$ and $t_0 \in (a, b)$. g is referred to as GH-differentiable at t_0, if there exists $g'_G(t_0) \in \mathcal{F}^n$ such that

$$g(t_0 + h) \ominus_H g(t_0), g(t_0) \ominus_H g(t_0 + h), \tag{43}$$

and

$$\lim_{h \to 0^+} \frac{g(t_0 + h) \ominus_H g(t_0)}{h} = \lim_{h \to 0^+} \frac{g(t_0) \ominus_H g(t_0 + h)}{h} = g'_G(t_0). \tag{44}$$

Example 3.5 [25] Consider the fuzzy mapping $g : \mathbb{R} \to \mathcal{F}'$ defined by $g(t) = C.t$, where C is a fuzzy number defined with its α-levels by $[C]^\alpha = [1 + \alpha, 2(3 - \alpha)t]$. Then,

$$[g(t)]^\alpha = \begin{cases} [1 + \alpha, 2(3 - \alpha)t], & \ge 0, \\ [2(3 - \alpha)t, 1 + \alpha], & t < 0. \end{cases} \tag{45}$$

Obviously, the functions g_l^α and g_r^α are not differentiable at $t = 0$. But g is GH-differentiable on R and $g'_G(t) = C$. That is, g is GH-differentiable at $t = 0$.

Let the following be the fuzzy initial value problem (FIVP):

$$\begin{cases} x'(t) = g(t, x(t)), & t \in J = [0, T], \\ x(0) = x_0, \end{cases} \tag{46}$$

with x' as derivative and is considered in the sense of GH-differentiable, where at the end points of J only one-sided derivative is considered, and the fuzzy function $g : J \times \mathcal{F}' \to \mathcal{F}'$ is continuous. The initial data x_0 in $\mathcal{F}'$. We denote $C^1(J, \mathcal{F}')$ the collections of all continuous fuzzy functions $g : J \to \mathcal{F}'$ with continuous derivative.

Lemma 3.6 [24] A fuzzy function $x \in C^1(J, \mathcal{F}')$ is a solution of (4.1) if and only if it verifies the integral equation

$$x(t) = x_0 \ominus_H (-1) \int_0^t g(s, x(s))ds, \quad t \in J = [0, T]. \tag{47}$$

Theorem 3.6 Suppose $g : J \times \mathcal{F}' \to \mathcal{F}'$ is continuous such that (i) $g(t, x) < g(t, y)$, for $x < y$, (ii) there exist some constants $\tau > 0$ large enough such that $\lambda \in \left(0, \frac{1}{2(\rho - \delta)}\right)$ and the metric for $x, y \in \mathcal{F}'$, with $x < y$ and $t \in J$ such that

$$||g(t, x(t)) - g(t, y(t)||_{\mathbb{R}} \le \tau \max_{t \in J} \left\{ d_\infty(x, y)e^{-\tau(t - \delta)} \right\}. \tag{48}$$

Then, (46) has a solution in $C^1(J, \mathcal{F}')$.

Proof Let $C^1(J, \mathcal{F}')$ be endowed with

$$d_\tau(x, y) = \sup_{t\in J} \max_{t\in J} \left\{ d_\infty(x(t), y(t)) e^{-\tau(t-\delta)} \right\}, \tag{49}$$

for $x, y \in C^1(J, \mathcal{F}')$ and $\tau > 0$. Then, with $g(x) = \ln(x), x > 0$ and $h = 0, (C^1(J, \mathcal{F}'), d_\tau)$ is $\mathcal{F}$ complete metric space.

Let $A, B : C^1(J, \mathcal{F}') \to (0, 1]$. For $x \in C^1(J, \mathcal{F}')$,

$$L_x(t) = x_0 \ominus_H (-1) \int_0^t g(s, x(s)) ds. \tag{50}$$

Let $x < y$. Then, it follows from the assumption of definition 3.1 (a) that

$$L_x(t) = x_0 \ominus_H (-1) \int_\delta^t g(s, x(s)) ds < x_0 \ominus_H (-1) \int_\delta^t g(s, y(s)) ds = R_y(t). \tag{51}$$

If $L_x(t) \neq R_y(t)$ and $T, f : C^1(J, \mathcal{F}') \to \mathcal{F}^{C^1(J,\mathcal{F}')}$ as

$$\beta_T x(r) = \begin{cases} A(x), & if\, r(t) = L_x(t), \\ 0, & otherwise, \end{cases} \tag{52}$$

$$\beta_f x(r) = \begin{cases} B(y), & if\, r(t) = L_y(t), \\ 0, & otherwise. \end{cases} \tag{53}$$

Again, if $\alpha_T(x) = A(x)$ and $\alpha_f(y) = B(y)$, we get

$$[Tx]_{\alpha_T(x)} = \{r \in X : (Tx)(t) \geq A(x)\} = \{L_x(t)\}, \tag{54}$$

Similarly, $[fy]_{\alpha_f(y)} = \{R_y(t)\}$

$$\begin{aligned}
&H([Tx]_{\alpha_T(x)}, [fy]_{\alpha_f(y)}) = \max \left\{ \begin{matrix} x \in [Tx]_{\alpha_T(x)}, & \sup\limits_{y\in[fy]_{\alpha_T(y)}} & inf ||x-y||_{\mathbb{R}} \\ y \in [fy]_{\alpha_f(y)}, & \sup\limits_{x\in[Tx]_{\alpha_f(x)}} & inf ||x-y||_{\mathbb{R}} \end{matrix} \right\} \leq \\
&\max \left\{ \sup_{t\in J} ||L_x(t) - R_y(t)||_{\mathbb{R}} \right\} = \sup_{t\in J} ||L_x(t) - R_y(t)||_{\mathbb{R}} = \sup_{t\in J} \left|\left| \int_\delta^t g(s, x(s)) ds - \int_0^t g(s, y(s)) ds \right|\right|_{\mathbb{R}} \\
&\leq \sup_{t\in J} \left\{ \int_\delta^t ||g(s, x(s)) - g(s, y(s))|| ds \right\} \leq \sup_{t\in J} \left\{ \int_\delta^t du \lambda \max\{D_\infty(x, y) e^{-\tau(t-\delta)}\} ds \right\} \leq \\
&\lambda \sup_{t\in J} \left\{ (t-\delta) \max\{D_\infty(x, y) e^{-\tau(t-\delta)}\} \right\} \leq \lambda(t-\delta) d_{\mathcal{F}}(x, y) \leq \frac{1}{2} d_{\mathcal{F}}(x, y) = \psi(d_{\mathcal{F}}(x, y)) \leq \\
&\phi(d_{\mathcal{F}}(x, y)) - \varphi \begin{pmatrix} d_{\mathcal{F}}(x, y), d_{\mathcal{F}}(x, [Tx]_{\alpha_T(x)}), d_{\mathcal{F}}(y, [fy]_{\alpha_f(y)}), \\ d_{\mathcal{F}}(x, [fy]_{\alpha_f(y)}), d_{\mathcal{F}}(y, [Tx]_{\alpha_T(x)}) \end{pmatrix}.
\end{aligned} \tag{55}$$

Hence, all the conditions of Theorem 2.1 are satisfied with $\phi(t) = 1/2t$ and $\varphi(t) = 1/4t$, for $t > 0$. Thus, x^* is a solution of (46).

Conclusion

This study demonstrates applicability of $\mathcal{F}$-metric spaces in establishing some common α-fixed point results for fuzzy enriched $\phi - \varphi$ contraction in a complete $\mathcal{F}$-metric spaces. This research offers significant boost in the understanding of $\mathcal{F}$-metric space, through illustrative examples, we retained the practical applicability of the theoretical framework and explored as an application, the existence of solution for fuzzy initial value problem of integro-differential equations in the context of generalized Hukuhara derivative. Future work could also explore the extension of this results to other types of fuzzy mappings.

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Авторы

Лакшми Ратхур, к.ф.-м.н., доцент, Национальный технологический институт, Чалтланг, Айджал 796 012, штат Мизорам, Индия.
E-mail: laxmirathour817@gmail.com

Винай Сингх, к.ф.-м.н., доцент, Национальный технологический институт, Чалтланг, Айджал 796 012, штат Мизорам, Индия.
E-mail: vinaysingh14aug@gmail.com

Мухаммед Раджи, к.ф.-м.н., доцент, Научно-технический университет Конфлюэнс, Осара, штат Коги, Нигерия.
E-mail: rajimuhammed11@gmail.com

Яхая Джибрин Данджума, к.ф.-м.н., доцент, Научно-технический университет Конфлюэнс, Осара, штат Коги, Нигерия.
E-mail: yahayajd@custech.edu.ng

Амину Ибрагим, к.ф.-м.н., доцент, Научно-технический университет Конфлюэнс, Осара, штат Коги, Нигерия.
E-mail: Ibrahima@custech.edu.ng

Лакшми Нараян Мишра, д.ф.-м.н., профессор, Технологический институт Веллуру, Веллуру, 632 014, штат Тамилнад, Индия.
E-mail: lakshminarayanmishra04@gmail.com

Вишну Нараян Мишра, д.ф.-м.н., профессор, Национальный университет малых народов им. Индиры Ганди, Лалпур, Амаркантак, Ануппур, штат Мадхья-Прадеш 484 887, Индия.
E-mail: vishnunarayanmishra@gmail.com

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Authors

Laxmi Rathour, Ph.D, Associate Professor, National Institute of Technology, Chaltlang, Aizawl 796 012, Mizoram, India.
E-mail: laxmirathour817@gmail.com

Vinay Singh, Ph.D, Associate Professor, National Institute of Technology, Chaltlang, Aizawl 796 012, Mizoram, India.
E-mail: vinaysingh14aug@gmail.com

Muhammed Raji, Ph.D, Associate Professor, Confluence University of Science and Technology, Osara, Kogi State, Nigeria.
E-mail: rajimuhammed11@gmail.com

Yahaya Jibrin Danjuma, Ph.D, Associate Professor, Confluence University of Science and Technology, Osara, Kogi State, Nigeria.
E-mail: yahayajd@custech.edu.ng

Aminu Ibrahim, Ph.D, Associate Professor, Confluence University of Science and Technology, Osara, Kogi State, Nigeria.
E-mail: Ibrahima@custech.edu.ng

Lakshmi Narayan Mishra, Ph.D, Professor, Vellore Institute of Technology, Vellore 632 014, Tamil Nadu, India.
E-mail: lakshminarayanmishra04@gmail.com

Vishnu Narayan Mishra, Ph.D, Professor, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India.
E-mail: vishnunarayanmishra@gmail.com

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