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СЖИМАЮЩИЕ ОТОБРАЖЕНИЯ ПО ЧИРИЧУ С ТОЧКИ ЗРЕНИЯ ТЕОРИИ ОТНОШЕНИЙ В $\mathcal{F}$-МЕТРИЧЕСКИХ ПРОСТРАНСТВАХ И РАЗВИТИЕ ТЕОРИИ НЕПОДВИЖНЫХ ТОЧЕК

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Вводится понятие $\mathcal{F}$-метрического пространства. Представлены результаты применения теории неподвижных точек и теории отношений для случая обогащенного ϕ-сжимающего отображения в полном $\mathcal{F}$-метрическом пространстве. Работа представляет собой существенный вклад в рассмотрение отображений в рамках теории отношений и в теорию неподвижных точек; представление результатов в виде численных примеров демонстрирует широту их теоретической и практической применимости. Также исследовано приложение описанного подхода к решению двухточечных краевых задач.

Ключевые слова: $\mathcal{F}$-метрическое пространство, неподвижная точка, бинарное отношение, теория отношений, обогащенное ϕ-φ сжимающее отображение, R-полнота.

ĆIRIĆ CONTRACTION TYPE OF RELATION-THEORETIC MAPPINGS IN $\mathcal{F}$-METRIC SPACES AND ADVANCEMENTS IN FIXED POINT APPROACH

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The article is aim to introduce the concept of $\mathcal{F}$-metric spaces and establish some fixed point results for relation-theoretic enriched ϕ-contraction in a complete $\mathcal{F}$-metric spaces. These advancements provide a significant contribution to the existing literature on relation-theoretic mapping and fixed point theory, broadening the theoretical and practical applicability of the findings are illustrative numerical examples. Also, we explore as an application, the solution for two points boundary value problems.

Keywords: $\mathcal{F}$-metric space, Fixed point, Binary relation, Relation-theoretic enriched ϕ-φ contraction mapping, R-completeness.

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Introduce

One of the most fascinating fields of research in the evolution of nonlinear analysis is fixed point theory. One of the fundamental results of fixed point theory, the Banach fixed point theorem [1], is crucial for demonstrating the existence and uniqueness of solutions to a wide range of mathematical problems. By extending the Banach fixed point theorem to the concept of order-theoretic fixed point outcomes, Turinici [2] extended Banach's work and provided a new approach to proving fixed point theorems. By developing an order-theoretic version of the Banach contraction principle and demonstrating its applicability to matrix equations, Ran and Reurings [3] adopted this strategy. Later, Samet and Turinici [4] developed fixed point solutions for nonlinear contractions based on the symmetric closure of an amorphous binary relation, furthering this line of research.

By extending the contraction principle introduced by Banach to arbitrary binary relations, Alam and Imdad [5] created a relation-theoretic fixed point theorem that unifies a number of previously known order-theoretic conclusions.

A new version of the Banach contraction principle for complete metric spaces with a binary relation was presented by Alam and Imdad [6]. In contrast to earlier fixed point theorems, their method relies on relation-theoretic analogues of contraction, completeness, and continuity rather than the conventional metrical definitions of these terms.

Moreover, Jleli and Samet [7] introduced a new metric space, which is referred to as $\mathcal{F}$-metric space, as an extension of the classical metric space. In this direction, Alnaser et al. [8] used the notion to establish some fixed point results in F-metric space with some open problems given, proffer fixed point theorems in L-fuzzy mappings. Recently, Tomar and Joshi [9] used the concept of $\mathcal{F}$-metric space as a generalization of traditional metric space and established fixed point results in the context of relation-theoretic contractions.

Motivated by the significance of the improvements, extensions and generalizations of recent and classical results in literature to the notion of an $\mathcal{F}$-metric space, we introduce the concepts of $\mathcal{F}$-metric spaces and subsequently establish some fixed point results relation-theoretic enriched ϕ-contraction mapping within the framework of complete $\mathcal{F}$-metric spaces. To bolster our findings, we offer illustrative numerical examples demonstrating the practical application of the presented results. Furthermore, we explore as an application, the solution for two points boundary value problems.

1. Preliminaries

In this section, we introduce the notion of $\mathcal{F}$-metric space and relation-theoretic enriched ϕ-contraction mapping.

Definition 1.1 [10] Let R be a nonempty set's binary relation. Reflexive X is defined as $(x,x) \in R\,\forall x \in X$. When $(x,y) \in R$ then $(y,x) \in R$, it is symmetric; when $(x,y) \in R$ and $(y,x) \in R$ then $x = y$; when $(x,y) \in R$ and $(y,z) \in R$ then $(x,z) \in R$, it is transitive; and when $[x,y] \in R\,\forall x,y \in X$, it is connected or complete or dichotomous.

Definition 1.2 [11] On a nonempty set X, let R be a binary relation. Next, R^s represents the symmetric closure of R, which is defined as the set $R \cup R^{-1}$ by

$$R^s := R \cup R^{-1}. \tag{1}$$

Definition 1.3 [12] On a nonempty set X, let R be a binary relation. R-preserving sequences are thus defined as $\{x_n\} \subset X$ if

$$(x_n, x_{n+1}) \in R \forall n \in N_0. \tag{2}$$

Definition 1.4 [13] Let f be a self-mapping on a set X that is not empty. Then, if for every $x, y \in X$, a binary relation R on X is T-closed,

$$(x, y) \in R \Rightarrow (Tx, Ty) \in R. \tag{3}$$

Definition 1.5 [14] Let (X, d) be a metric space and R be a binary relation on a nonempty set X. If all of the R-preserving Cauchy sequences in X converge, then (X, d) is R-complete.

Definition 1.6 [15] Consider a binary relation and a metric space (X, d). When a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in R \forall k \in N_0$ exists for each R-preserving sequence $\{x_n\}$ such that $x_n \xrightarrow{d} x$, then R on a nonempty set X is said to be d-self-closed.

Definition 1.7 [16] On a nonempty set X, let R be a binary relation. If there is a $z \in X$ such that $(x, z) \in R$ and $(y, z) \in R$ for every pair $x, y \in E$, then the subset E of X is said to be R-directed.

Definition 1.8 [2] A path of length $k \in N$ from x to y in R is defined as a finite sequence $\{r_0, r_1, r_2, ..., r_k\}$ in X if $r_0 = x$, $r_k = y$ and $(r_i, r_{i+1}) \in R$, $0 \leq i \leq k-1$.

Definition 1.9 [10, 12] If a mapping $\phi : [0, \infty) \to [0, \infty)$ satisfies the following conditions, it is called a comparison function: (i) ϕ is increasing, (ii) $\sum_{n=1}^{+\infty} \phi^n(t) < \infty$ for each $t > 0$.

Definition 1.10 [17, 18] If there is a function $\phi : [0, \infty) \to [0, \infty)$, then a function T from a metric space (X, d) onto itself is called ϕ-contraction fulfilling

$$d(Tx, Ty) \leq \phi(d(x, y)), \quad \forall x, y \in X. \tag{4}$$

$\mathcal{F}$-metric space is now introduced as follows: let $g : (0, +\infty) \to \mathbb{R}$ and $\mathcal{F}$ referred to the collection of functions g verifying: $(\mathcal{F}_1) 0 < x < t \Rightarrow g(x) \leq g(t)$, $(\mathcal{F}_2)$ for the sequence $\{x_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} x_n = 0 \Leftrightarrow \lim_{n\to\infty} g(x_n) = -\infty$.

Definition 1.11 [3, 19] Let X be nonempty set and $d_{\mathcal{F}} : X \times X \to [0, +\infty)$. Assume there exists $(g, h) \in \mathcal{F} \times [0, +\infty)$ such that (i) $(x, y) \in X \times X, d_{\mathcal{F}}(x, y) = 0 \Leftrightarrow x = y$, (ii) $d_{\mathcal{F}}(x, y) = d_{\mathcal{F}}(y, x)$, for all $(x, y) \in X \times X$, (iii) for each $(x, y) \in X \times X$, for each $N \in N$, $N \geq 2$, and for each $(u_i)_{i=1}^N \subset X$ with $(u_1, u_N) = (x, y)$, then

$$d_{\mathcal{F}}(x, y) > 0 \Rightarrow g(d_{\mathcal{F}}(x, y)) \leq g\left[\sum_{i=1}^{N-1} d_{\mathcal{F}}(x_i, x_{i+1})\right] + h. \tag{5}$$

Then, $d_{\mathcal{F}}$ is referred to as $\mathcal{F}$-metric on X and $(X, d_{\mathcal{F}})$ is referred to as $\mathcal{F}$-metric space.

Example 1.12 [3] Let $d_{\mathcal{F}} : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ be a function define by

$$d_{\mathcal{F}} = \begin{cases} (x-y)^2, & if\,(x, y) \in [0, 3] \times [0, 3], \\ |x-y|, & if\,(x, y) \notin [0, 3] \times [0, 3], \end{cases} \tag{6}$$

with $g(t) = \ln(t)$ and $h = \ln(3)$ admits $\mathcal{F}$-metric.

Definition 1.13 [10] Let $(X, d_{\mathcal{F}})$ be $\mathcal{F}$-metric space. (i) Let $\{x_n\} \subseteq X$. The sequence $\{x_n\}$ is referred to as $\mathcal{F}$-convergent to $x \in X$ if $\{x_n\}$ is convergent to x in $\mathcal{F}$-metric $d_{\mathcal{F}}$. (ii) The sequence $\{x_n\}$ is referred to as $\mathcal{F}$-Cauchy, if $\lim_{n,m\to\infty} d_{\mathcal{F}}(x_n, x_m) = 0$. (iii) $(X, d_{\mathcal{F}})$ is $\mathcal{F}$-complete if each $\mathcal{F}$-Cauchy sequence in X is $\mathcal{F}$-convergent to $x \in X$.

2. Main results

Theorem 2.1 Let $(X, d_{\mathcal{F}}, R)$ remains as a relational metric space endued with $\mathcal{F}$-metric space. Assume that the self mapping $T : X \to X$ satisfies the assertions: (i) $T(X) \subseteq Y \subseteq X$ so that $(Y, d_{\mathcal{F}})$ is R-complete, (ii) $X[T, R]$ is nonempty, (iii) R is termed as T-closed; (iv) T remains R-continuous, or $R|_Y$ is regarded as $d_{\mathcal{F}}$-self-closed; (v) for all $(x, y) \in R$, there exist $\phi \in \Psi$ verifying

$$d_{\mathcal{F}}(Tx, Ty) \leq \phi(M(x, y)), \tag{7}$$

where

$$M(x,y) = \max\left\{\begin{matrix} d_{\mathcal{F}}(x,y), d_{\mathcal{F}}(x,Tx), d_{\mathcal{F}}(y,Ty), \dfrac{d_{\mathcal{F}}(x,Tx)d_{\mathcal{F}}(y,Ty)}{d_{\mathcal{F}}(x,y)+d_{\mathcal{F}}(x,Ty)+d_{\mathcal{F}}(y,Tx)}, \\ \dfrac{d_{\mathcal{F}}(x,Tx)d_{\mathcal{F}}(x,Ty)+d_{\mathcal{F}}(y,Tx)d_{\mathcal{F}}(y,Ty)}{d_{\mathcal{F}}(y,Tx)+d_{\mathcal{F}}(x,Ty)} \end{matrix}\right\}, \tag{8}$$

then T admits a fixed point in X. Furthermore, if (vi) $T(X)$ remains R^s-connected, then T fulfils a unique fixed point in X. Additionally, the sequence $\{x_n\} \subseteq X$, $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$, then $\mathcal{F}$-convergent to a fixed point.

Proof Using the condition (ii), there exist $x_0 \in X(T, R)$ and construct sequence $\{x_n\} \subset X$ satisfying

$$x_{n+1} = Tx_n, \ \forall n \in \mathbb{N}. \tag{9}$$

Since $(x_0, Tx_0) \in R$ and T-closedness of R, we have

$$(Tx_0, T^2x_0), (T^2x_0, T^3x_0), ..., (T^nx_0, T^{n+1}x_0) \in R. \tag{10}$$

Using (9) and (10), we have

$$(x_n, x_{n+1}) \in R, \ \forall n \in \mathbb{N}. \tag{11}$$

Hence, $\{x_n\}$ is R-preserving sequence. Using the (8), (9) and (11), we have

$$d_{\mathcal{F}}(x_n, x_{n+1}) = d_{\mathcal{F}}(Tx_{n-1}, Tx_n) \leq \phi(N(x_{n-1}, x_n)), \tag{12}$$

where

$$\begin{aligned} N(x_{n-1},x_n) &= \max\left\{\begin{matrix} d_{\mathcal{F}}(x_{n-1},x_n), d_{\mathcal{F}}(x_{n-1},Tx_{n-1}), d_{\mathcal{F}}(x_n,Tx_n), \\ \dfrac{d_{\mathcal{F}}(x_{n-1},Tx_{n-1})d_{\mathcal{F}}(x_n,Tx_n)}{d_{\mathcal{F}}(x_{n-1},x_n)+d_{\mathcal{F}}(x_{n-1},Tx_n)+d_{\mathcal{F}}(x_n,Tx_{n-1})}, \\ \dfrac{d_{\mathcal{F}}(x_{n-1},Tx_{n-1})d_{\mathcal{F}}(x_{n-1},Tx_n)+d_{\mathcal{F}}(x_n,Tx_{n-1})d_{\mathcal{F}}(x_n,Tx_n)}{d_{\mathcal{F}}(x_n,Tx_{n-1})+d_{\mathcal{F}}(x_{n-1},Tx_n)} \end{matrix}\right\} = \\ &= \max\left\{\begin{matrix} d_{\mathcal{F}}(x_{n-1},x_n), d_{\mathcal{F}}(x_{n-1},x_n), d_{\mathcal{F}}(x_n,x_{n+1}), \\ \dfrac{d_{\mathcal{F}}(x_{n-1},x_n)d_{\mathcal{F}}(x_n,x_{n+1})}{d_{\mathcal{F}}(x_{n-1},x_n)+d_{\mathcal{F}}(x_{n-1},x_{n+1})+d_{\mathcal{F}}(x_n,x_n)}, \\ \dfrac{d_{\mathcal{F}}(x_{n-1},x_n)d_{\mathcal{F}}(x_{n-1},x_{n+1})+d_{\mathcal{F}}(x_n,x_n)d_{\mathcal{F}}(x_n,x_{n+1})}{d_{\mathcal{F}}(x_n,x_n)+d_{\mathcal{F}}(x_{n-1},x_{n+1})} \end{matrix}\right\} = \\ &= \max\left\{\begin{matrix} d_{\mathcal{F}}(x_{n-1},x_n), d_{\mathcal{F}}(x_{n-1},x_n), \\ d_{\mathcal{F}}(x_n,x_{n+1}, d_{\mathcal{F}}(x_{n-1},x_n), d_{\mathcal{F}}(x_{n-1},x_n)) \end{matrix}\right\} \end{aligned} \tag{13}$$

Implies

$$d_{\mathcal{F}}(x_n, x_{n+1}) \leq \phi\left(\max\left\{\begin{matrix} d_{\mathcal{F}}(x_{n-1},x_n), d_{\mathcal{F}}(x_{n-1},x_n), \\ d_{\mathcal{F}}(x_n,x_{n+1}, d_{\mathcal{F}}(x_{n-1},x_n), d_{\mathcal{F}}(x_{n-1},x_n)) \end{matrix}\right\}\right) \tag{14}$$

Now suppose $d_{\mathcal{F}}(x_n, x_{n+1}) > d_{\mathcal{F}}(x_{n-1}, x_n)$. Using (i) of definition 1.9, we have

$$d_{\mathcal{F}}(x_n, x_{n+1}) \leq \phi(d_{\mathcal{F}}(x_{n-1}, x_n)) < d_{\mathcal{F}}(x_n, x_{n+1}), \tag{15}$$

a contradiction. So

$$d_{\mathcal{F}}(x_n, x_{n+1}) \leq \phi(d_{\mathcal{F}}(x_{n-1}, x_n)). \tag{16}$$

By continuing the process on n and using (ii) of definition 1.11, we have

$$d_{\mathcal{F}}(x_n, x_{n+1}) \leq \phi^n(d_{\mathcal{F}}(x_0, x_1)), n \in \mathbb{N}_0. \tag{17}$$

Using (17), $\forall m,\, n \in \mathbb{N}_0$ and $m \geq n$, we get

$$\sum_{i=n}^{m-1} d_{\mathcal{F}}(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \phi^i(d_{\mathcal{F}}(x_0, x_1)). \tag{18}$$

By (ii) of definition 1.11 and as $n \to \infty$, $\sum_{i=n}^{m-1} \phi^i(d_{\mathcal{F}}(x_0, x_1)) \to 0$. That is, there exists $\epsilon > 0$ fulfilling $0 < \sum_{i=n}^{m-1} \phi^i(d_{\mathcal{F}}(x_0, x_1)) < \epsilon$.

Let (g, h) fulfil (iii) of definition 1.8. By $(\mathcal{F}_2)$, for fixed $\epsilon > 0$,

$$g\left(\sum_{i=n}^{m-1} \phi^i(d_{\mathcal{F}}(x_0, x_1))\right) < g(\epsilon) - h. \tag{19}$$

Using (iii) of definition 1.8 and (19), $d_{\mathcal{F}}(x_m, x_n) > 0$.

Implies

$$g(d_{\mathcal{F}}(x_n, x_m)) \leq g\left(\sum_{i=n}^{m-1} d_{\mathcal{F}}(x_i, x_{i+1})\right) + h \leq g\left(\sum_{i=n}^{m-1} \phi^i(d_{\mathcal{F}}(x_0, x_1))\right) + h < g(\epsilon). \tag{20}$$

Again, by $(\mathcal{F}_1)$

$$d_{\mathcal{F}}(x_n, x_m) < \epsilon,\ m \geq n. \tag{21}$$

Thus, the $\mathcal{F}$-Cauchy sequence in X is $\{x_n\}$. Hence, $\{x_n\}$ is an $\mathcal{F}$-Cauchy sequence that preserves R. $\{x_n\}$ is also an R-preserving $\mathcal{F}$-Cauchy sequence in Y since $\{x_n\} \subseteq T(X) \subseteq Y$. The hypothesis (i) states that if $(Y, d_{\mathcal{F}})$ is R-complete, then $x_n \overset{d_{\mathcal{F}}}{\to} d_{\mathcal{F}} x^*$.

If T is R-continuous, we have

$$x_{n+1} = Tx_n \overset{d_{\mathcal{F}}}{\to} d_{\mathcal{F}})Tx^*. \tag{22}$$

So $Tx^* = x^*$, shows x^* is a fixed point of T. However, if $R|_Y$ is $d_{\mathcal{F}}$-self-closed, then given the R-preserving sequence $\{x_n\}$ in Y as $x_n \overset{d_{\mathcal{F}}}{\to} d_{\mathcal{F}} x^*$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x^*] \in R|_Y \subseteq R$, $\forall k \in N_0$ and $x_{n_{k+1}} \overset{d_{\mathcal{F}}}{\to} d_{\mathcal{F}} x^*$.

Now

$$g(d_{\mathcal{F}}(x^*, Tx^*)) \leq g(d_{\mathcal{F}}(x^*, Tx_{n_k}) + d_{\mathcal{F}}(Tx_{n_k}, Tx^*)) + h = g(d_{\mathcal{F}}(x^*, x_{n_{k+1}}) + d_{\mathcal{F}}(x_{n_{k+1}}, Tx^*)) + h \to -\infty. \tag{23}$$

As $n \to \infty$, using $(\mathcal{F}_2)$, a contradiction. Thus, $d_{\mathcal{F}}(x^*, Tx^*) = 0$, that is, $x^* = Tx^*$. Implies x^* is a fixed point of T.

Let u and v be two different fixed points of T such that $Tu = u$ and $Tv = v$ in order to demonstrate the fixed point's uniqueness. Thus

$$u, v \in T(X), \quad T^n u = u, T^n v = v, n \in \mathbb{N}_0. \tag{24}$$

Given that $T(X)$ is R^s-connected, there is a path say $\{z_1, z_2, z_3, ..., z_k\}$ of finite length k in R^s-connected from u to v, such that $z_0 = u$, $z_k = v$ and $(z_i, z_{i+1}) \in R^s$, $i = 0, 1, 2, ..., k-1$. R is T-closed according to hypothesis (iii), and $(T^n z_i, T^n z_{i+1}) \in R^s$, where $i = 0, 1, 2, ..., k-1$ and $n \in \mathbb{N}_0$. Assuming that $d_{\mathcal{F}}(u, v) > 0$,

$$g(d_{\mathcal{F}}(u, v)) = g(d_{\mathcal{F}}(T^n z_0, T^n z_k)) \leq g\left(\sum_{i=0}^{k-1} d_{\mathcal{F}}(T^n z_i, T^n z_{i+1})\right) + h \leq$$

$$g\left(\mu \sum_{i=0}^{k-1} d_{\mathcal{F}}(T^{n-1} z_i, T^{n-1} z_{i+1})\right) + h \leq g\left(\mu^2 \sum_{i=0}^{k-1} d_{\mathcal{F}}(T^{n-2} z_i, T^{n-2} z_{i+1})\right) + h. \tag{25}$$

Continuing this process, we have

$$\leq g\left(\mu^n \sum_{i=0}^{k-1} d_{\mathcal{F}}(z_i, z_{i+1})\right) + h \to -\infty, \tag{26}$$

as $n \to \infty$, using $(\mathcal{F}_2)$, a contradiction. Thus, $d_{\mathcal{F}}(u,v) = 0$, implies, $u = v$.

Example 2.2. Let $X = (-10, 10]$ be the set on $\mathcal{F}$-metric defined by

$$d_{\mathcal{F}}(x,y) = \begin{cases} 2^{|x-y|}, & x \neq y, \\ 0, & x = y, \end{cases} \tag{27}$$

with $g(t) = -1/t$ and $h = 1$. The relation R on X is define as $R(x,y) = x, y \in X : x \geq y$ with closure symmetry as $R^s = \{(x,y) : \textit{either}\, x \leq y \textit{ or}\, y \leq x\}$. Let $\phi : [0,\infty) \to [0,\infty)$ be define as $\phi(t) = 4/5t$ and $T : X \to X$ by

$$T(x) = \begin{cases} -3/2, & p \in (-10,-1] \\ -1, & p \in (-1, 0] \\ -1/2, & p \in (0, 2] \\ \ \ 2, & p \in (2, 10] \end{cases} \tag{28}$$

If $Y = [2,5) \subseteq X, T(X) = [-3/2, -1, -1/2, 2]$. Then $(Y, d_{\mathcal{F}})$ is R-complete and $T(X) \subseteq Y \subseteq X$. However, X is not R-complete or complete. It is evident that T is neither continuous nor R-continuous, and that R is T-closed. A sequence that is R -preserving and converges to 0 is $x_n = 1/n$, while $\{x_n\} = \{-1/2\}$ converges to $-1/2$. $T(X)$ is R^s-connected, and $R|_Y$ is $d_{\mathcal{F}}$-self-closed. Assume that $x = -3/2$ and that the fixed point is unique at $-3/2$ in order to verify this. Furthermore, $\mathcal{F}$-converges to $-3/2$ if the sequence $\{x_n\} \subseteq X$, such that $x_n = -3n/(2n+1)$, $n \in \mathbb{N}$.

Let $(X, d_{\mathcal{F}}, R)$ remains as a relational metric space endued with $\mathcal{F}$-metric space. Assume that the self mapping $T : X \to X$ satisfies the assertions: (i) $T(X) \subseteq Y \subseteq X$ so that $(Y, d_{\mathcal{F}})$ is R-complete, (ii) $X[T,R]$ is nonempty, (iii) R is termed as T-closed; (iv) T remains R-continuous, or $R|_Y$ is regarded as $d_{\mathcal{F}}$-self- closed; (v) for all $(x,y) \in R$, there exist $\phi \in \Psi$ verifying

$$d_{\mathcal{F}}(Tx, Ty) \leq \phi(N(x,y)), \tag{29}$$

where

$$N(x,y) = \max\left\{\begin{array}{l} d_{\mathcal{F}}(x,y), d_{\mathcal{F}}(x,Tx), d_{\mathcal{F}}(y,Ty), \dfrac{d_{\mathcal{F}}(x,Tx)d_{\mathcal{F}}(y,Ty)}{d_{\mathcal{F}}(x,y) + d_{\mathcal{F}}(x,Ty) + d_{\mathcal{F}}(y,Tx)}, \\ \dfrac{d_{\mathcal{F}}(x,Tx)d_{\mathcal{F}}(x,Ty) + d_{\mathcal{F}}(y,Tx)d_{\mathcal{F}}(y,Ty)}{d_{\mathcal{F}}(y,Tx) + d_{\mathcal{F}}(x,Ty)} \end{array}\right\}, \tag{30}$$

then T admits a fixed point in X.

Corollary 2.3. If (vi) $R|_{T(X)}$ remains complete, then T fulfils a unique fixed point in X. Moreover, if the sequence $\{x_n\} \subseteq X$, $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$, is $\mathcal{F}$-convergent to a fixed point

Proof. Theorem 2.1 is further supported by the fact that $x, y \in T(X)$, $[x,y] \in R|_T(X)$ indicates that x, y is a path of length 1 in $R^s|_{T(X)}$ from x to y if hypothesis (vi) is true. $T(X)$ is hence R^s-connectedness. Thus, we have the result.

Corollary 2.4. If (vi) $T(X)$ remains R^s-directed, then T fulfils a unique fixed point in X. Moreover, if the sequence $\{x_n\} \subseteq X$, $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$, is $\mathcal{F}$-convergent to a fixed point.

Proof. To further support the proof of Theorem 2.1, if the hypothesis (vi) is true, then $x, y \in T(X)$, $existsz \in X$ such that $(x,z) \in R^s$ and $(y,z) \in R^s$ indicates that x, y, z is a path of length 2 in R^s from x to y. $T(X)$ is R^s-connected as a result. We thus have the result.

Corollary 2.5. if (vi) $\mu(x, y, R^s) \neq \emptyset$, then T fulfils a unique fixed point in X. Moreover, if the sequence $\{x_n\} \subseteq X$, $x_{n+1} = Tx_n$, $n \in \mathbb{N}_0$, is $\mathcal{F}$-convergent to a fixed point

Proof. As an extension of the proof of Theorem 2.1, if the hypothesis (vi) is true, then $x, y \in T(X)$, and there is a path from x to y, say $\{z_1, z_2, z_3, ..., z_k\}$ of finite length k in R^s, such that $z_0 = x$, $z_k = y$. $T(X)$ is hence R^s-connected. Thus, we have the result.

3. Application

Here, we solve the boundary value problems by applying our previous findings. Assume that $X = C[I, R]$ and that $I[0, 1]$ and $C[I, R]$ are the collections of all continuous functions on $[0, 1]$. We now consider the subsequent Theorem.

Theorem 3.1. Let the second order differential equation be

$$\frac{d^2y}{dx^2} = \phi(t, x(t)), \quad \textit{with } x(0) = 0,\ x(1) = 0, \tag{31}$$

where $\phi : [0, 1] \times \mathbb{R} \to \mathbb{R}$ represents a continuous function and $t \in I$.

Then, there exists $\lambda \geq 0$ such that

$$0 \leq \phi(t, x(t)) + \lambda x(t) - [\phi(t, y(t)) + \lambda y(t)] \leq \lambda \frac{x(t) - y(t)}{1 + x(t) - y(t)}, \tag{32}$$

where $x(t) \geq y(t)$ and $x(t)$, $y(t) \in X$, then x^* in X is the unique solution of (46).

Proof. The boundary value problem in (46) can be represented as

$$x''(t) + \lambda x(t) = \phi(t, x(t)) + \lambda x(t), \quad t \in [0, 1],\ x(0) = 0,\ x(1) = 0, \tag{33}$$

or

$$x(t) = \int_0^1 G(t, \xi)[\phi(\xi, x(\xi)) + \lambda x(\xi)]d\xi, \quad \textit{for } t \in I, \tag{34}$$

with the Green function as

$$G(t, \xi) = f(x) = \begin{cases} (1 - t)\xi, & 0 \leq \xi \leq t \leq 1, \\ (1 - \xi)t, & 0 \leq t \leq \xi \leq 1. \end{cases} \tag{35}$$

If we define a map $T : X \to X$, then $x \in X$ is a solution of (34) if it is the solution of (46). Then

$$Tx(t) = \int_0^1 G(t, \xi)[\phi(\xi, x(\xi)) + \lambda x(\xi)]d\xi,$$

with binary relation

$$R = \{(x, y) \in X \times X : x(t), y(t) \geq 0 \textit{ where } (x - y)(t) \geq 0, \forall t \in I\}. \tag{36}$$

Then, the following cases arises:

Case I. Now consider

$$d_{\mathcal{F}}(x(t), y(t)) = \begin{cases} \exp\left(\sup\limits_{t\in I} |x(t) - y(t)|\right), & x(t) \neq y(t), \\ 0, & x(t) = y(t), \end{cases} \tag{37}$$

where $g(t) = -1/t$, $t > 0$ and $h = 1$. Then, $(X, d_{\mathcal{F}})$ admits R-complete $\mathcal{F}$-metric spaces.

Case II. Let $\{x_n(t)\}$ be an R-preserving Cauchy sequence such that $x_n(t) \to x(t)$. In that case, $x_n(t) \geq x_{n+1}(t)$, $t \in I$, $n \in \mathbb{N}_0$, and $x_n(t)x_{n+1}(t) \geq 0$. Then, either $x_n(t) \leq 0$ or $x_n(t) \geq 0$ are present. We get a sequence of positive values that converges to $x(t)$ if we assume that $x(t) \geq 0$ for $t \in I$. Because of this, $x(t) \geq 0$, or $(x_n(t), x(t)) \in R$, $t \in I$, $n \in \mathbb{N}_0$. Implies that $d_{\mathcal{F}}$-self-closed is admitted by R.

Case III. For $(x, y) \in R$, that is, $x(t) \geq y(t)$ by (32), $\phi(t, y(t)) + \lambda y(t) \leq \phi(t, x(t)) + \lambda x(t)$, $\forall t \in I$ and $G(t, \xi) \geq 0$, $(t, \xi) \in I \times I$, we obtain

$$Tx(t) = \int_0^1 G(t, \xi)[\phi(\xi, x(\xi)) + \lambda x(\xi)]d\xi \geq \int_0^1 G(t, \xi)[\phi(\xi, y(\xi)) + \lambda y(\xi)]d\xi = Ty(t). \tag{38}$$

Implies $(Tx(t), Ty(t)) \in R$, that is, R is T-closed. For any $x(t) \geq 0$, $Tx(t) \geq 0$, that is, $(x(t), Tx(t)) \in R$. Then, $X[T, R]$ remains non-empty.

Case IV. For $(x, y) \in R$

$$\begin{aligned}
& d_{\mathcal{F}}(Tx(t), Ty(t)) = \exp\left(\sup_{t\in I}|Tx(t) - Ty(t)|\right) = \\
& \exp\left(\sup_{t\in I}|\int_0^1 G(t,\xi)[\phi(\xi, x(\xi)) + \lambda x(\xi)]d\xi - \int_0^1 G(t,\xi)[\phi(\xi, y(\xi)) + \lambda y(\xi)]d\xi\right) \\
& \leq \exp\left(\sup_{t\in I}|\int_0^1 G(t,\xi)\lambda\frac{y(\xi) - x(\xi)}{1 + y(\xi) - x(\xi)}d\xi\right) \leq \exp\left(\sup_{t\in I}|x(\xi) - y(\xi)|\int_0^1 G(t,\xi)d\xi\right) \\
& = \exp\left(\sup_{t\in I}|x(\xi) - y(\xi)|\int_0^1 (1-t)\xi d\xi + \int_0^1 (1-\xi)t d\xi\right) \leq \exp\left(\sup_{t\in I}|x(t) - y(t)|\frac{1}{8}\right) \\
& \leq \phi \exp\sup_{t\in I}|x(t) - y(t)| = \phi(N(x, y))
\end{aligned} \tag{39}$$

If $X = Y = C[I, R]$, Y admits that it is R^s-connected. Therefore, T has a unique fixed point and all of the hypotheses of Theorem 2.1 are satisfied.

Conclusion

In this paper, we explored the concept of $\mathcal{F}$-metric spaces to establish some fixed point results for relation-theoretic enriched ϕ-contraction utilizing a complete $\mathcal{F}$-metric spaces. This study provides significant advancements in the understanding of relation-theoretic mapping, using illustrative numerical examples, we showcased the practical applicability of the results and explored as an application, the solution for two points boundary value problems. Future work could also explore the extension of this results to quasimetric space and fuzzy set valued mappings.

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