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# ДВИЖЕНИЕ ТЕНЗОРОВ ПЛОТНОСТИ ВРАЩЕНИЯ СОГЛАСНО АФФИННО-МЕТРИЧЕСКОЙ ТЕОРИИ ГРАВИТАЦИИ

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Выведены уравнения движения для различных объектов в рамках аффинно-метрической теории гравитации. Данный класс теорий разрабатывается для расширения общей теории относительности и применения ее для изучения собственных свойств материи в присутствии гравитационного поля. Объекты, описываемые данным классом теорий, представлены как структуры, состоящие из траекторий и уравнений отклонения от траектории. При этом необходимо учитывать, что данный класс теорий относится к чисто калибровочным теориям гравитации.

*Ключевые слова*: метрическая аффинная гравитация, тензор плотности спина, калибровочная теория.

# MOTION OF SPINNING DENSITY TENSOR IN A METRIC AFFINE THEORY OF GRAVITY

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Equations of motion for different objects within the context of the metric affine theories of gravity (MAG) are derived. The classes of MAG are the developed version of General relativity cable to examine the intrinsic properties of matter in the presence of a gravitational field. Objects subjected to these classes are expressed as a structure made of paths and path deviation equations. It is vital to consider that MAG-type theories are classified as pure gauge theories of gravity

*Keywords*: metric affine gravity, spin density tensor, gauge theory.

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## Introduction

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This work aims to obtain path and path deviation equations in a class of metric affine gravity (MAG): Such geometry in which it contains nonvanishing curvature and torsion and non-metricity tensor simultaneously. These paths are analogously obtained in Riemannian geometry using the modified Bazanski Lagrangian (2006).

The problem of rotating objects has shown the status of tetrad vectors—the building blocks of teleparallel theories of gravity for rotated and parallelly transported, preserving GCT and LLT. The emergence of local Lorentz transformation may give rise to obtaining an invariant gravitational theory under gauge transformation through its corresponding spin connection (Lorentz Connection). Modern physics demands revising GR to be invariant under gauge transformation [6]. Accordingly, one may find out GR is written in terms of tetrad formalism. Likewise such a principle is existed in case of MAG type theories. The paper is organized as follows : Section 1 displays a glimpse of the underlying geometry of MAG. Section 2 discusses The path and path deviation equations fir objects defined in MAG. Section 3 shows the relationship between spinning density and spin tensor as an extension to The Weyesenhoff tensor. Section 4 performs the Papapetrou-like equations for spinning objects in MAG. Section 5 discusses the different consequences resulted from obtaining different path and path devation equations and the effective role of non-metricity tensor on different path and path deviation equations.

## 1. Underlying Geometrty of MAG

A metric affine thepry of Gravity is defined by a triplet $M, g_{\mu\nu}, \Gamma^{\mu}_{\nu\rho}$ where $M$ is a four dimensional space-time manifold, $g_{\mu\nu}$ is a metric tensor and $\Gamma^{\mu}_{\nu\rho}$ is the affine connection defined in the following way [7]:

$$\Gamma^{\mu}_{\nu\rho} \stackrel{\text{def}}{=} \left\{ {\mu \atop \nu\rho} \right\} + K^{\mu}_{\nu\rho} + L^{\mu}_{\nu\rho}, \tag{1}$$

where $\left\{ {\mu \atop \nu\rho} \right\}$ is the Christoffel symbol , $K^{\mu}_{\nu\rho}$ is the contortion of space-time and $L^{\mu}_{\nu\rho}$ is disformation tensor defined as follows:

$$\left\{ {\mu \atop \nu\rho} \right\} \stackrel{\text{def}}{=} \frac{1}{2} g^{\mu\delta}(g_{\delta\nu,\rho} - g_{\rho\delta,\nu} - g_{\nu\rho,\delta}), \tag{2}$$

$$K^{\mu} - \nu\rho \stackrel{\text{def}}{=} \frac{1}{2} g^{\mu\delta}(\Lambda_{\delta\nu\rho} - \Lambda_{\rho\delta\nu} - \Lambda_{\nu\delta\nu}), \tag{3}$$

and

$$L^{\mu}_{\nu}\rho \stackrel{\text{def}}{=} \frac{1}{2} g^{\mu\delta}(\Omega_{\delta\nu\rho} - \Omega_{\rho\delta\nu} - \Omega_{\nu\delta\nu}), \tag{4}$$

such that $\Lambda_{\delta\nu\rho}$ is the torsion tensor and $\Omega_{\delta\nu\rho}$ is the non-metricity tensor. In MAG theory it can be found that the main three geometrical objects are the curvature of space-time $R^{\mu}_{\nu\rho\sigma}$ , the torsion tensor and the non-metricity tensor expressed as follows:

$$R^{\mu}_{\nu\rho\sigma} = \Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\mu}_{\nu\sigma,\rho} + \Gamma^{\delta}_{\nu\rho}\Gamma^{\mu}_{\delta\sigma} - \Gamma^{\delta}_{\nu\sigma}\Gamma^{\mu}_{\delta\rho}, \tag{5}$$

$$\Lambda^{\mu}_{\nu\rho} = \Gamma^{\mu}_{\nu\rho} - \Gamma^{\mu}_{\nu\rho}, \tag{6}$$

and

$$\Omega_{\mu\nu\rho} = g_{\nu\rho||\mu} \neq 0, \tag{7}$$

in which

$$g_{\nu\rho||\mu} = g_{\nu\rho,\mu} - \Gamma^{\delta}_{\nu\mu} g_{\delta\rho} - \Gamma^{\delta}_{\nu\mu} g_{\delta\nu} \neq 0. \tag{8}$$

### 1.1. The Tetrad Formalism of Trinity Theory of Gravity

The structure of this space is defined completely by a set of n-contravariant vector fields $\underset{i}{\lambda}^{\mu}$ where $i = 1, 2, 3, ..., n)$ denotes tangent bundle vector , and $\mu (= 1, 2, 3, ..., n)$ denotes $\underset{i}{\lambda}_{\mu}$ of the vectors $\underset{i}{\lambda}^{\mu}$, in the determinant $|| \underset{i}{\lambda}^{\mu} ||$, is defined such that

$$\underset{i}{\lambda}^{\mu} \underset{j}{\lambda}_{\mu} = \delta_{ij}, \quad \underset{i}{\lambda}^{\mu} \underset{i}{\lambda}_{\nu} = \delta^{\mu}_{\nu}. \tag{9}$$

Using these vectors, the following second order symmetric tensors are defined:

$$g^{\mu\nu} \stackrel{\text{def}}{=} \underset{i}{\lambda}^{\mu} \underset{i}{\lambda}^{\nu}, \quad g_{\mu\nu} \stackrel{\text{def}}{=} \underset{i}{\lambda}_{\mu} \underset{i}{\lambda}_{\nu}. \tag{10}$$

and

$$\eta^{ij} \stackrel{\text{def}}{=} \lambda^{i}_{\mu}\lambda^{j\mu}, \eta_{ij} \stackrel{\text{def}}{=} \underset{i}{\lambda}_{\mu} \underset{j}{\lambda}^{\mu}. \tag{11}$$

**The Tetrad Condition in GR (i)**. It is well known that general relativity may be a gauge theory of gravity as being expressed tetrad vector invariant under general coordinate and Local Lorentz transformations if and only if

$$\underset{i}{\lambda}^{\nu}_{;\mu} = 0, \tag{12}$$

where $;^{\mu}$ is the covariant derivative defined by the Christoffel symbol, such that :

$$\left\{ {\alpha \atop \mu\nu} \right\} = \underset{i}{\lambda}^{\alpha}(\underset{i}{\lambda}_{\mu,\nu} + \omega^{\cdot j \cdot}_{i \cdot\cdot \nu} \underset{j}{\lambda}^{\alpha}). \tag{13}$$

The curvature tensor is defined to be

$$\bar{R}^{i}_{j\gamma\delta} = \omega^{i}_{j\gamma,\delta} - \omega^{i}_{j,\delta\gamma} + \omega^{k}_{j\gamma}\omega^{i}_{k\delta} - \omega^{k}_{j\delta}\omega^{i}_{k\gamma}. \tag{14}$$

Using the tetrad condition we can relate the curvature of space-time together with torsion of space time with microscopic structure due to the appearance of equations of motion combining holonomic (Greek letters) and annholonomic (Latin letters) coordinates. It can be found that $\bar{R}^{i}_{j\gamma\delta}$ the curvature of space time in (14) is expressed in terms of spin connection in which the latter indicates the existence of matter. From this perspective it is crystal clear to indicate that $\underset{i}{\lambda}^{\mu}$ and $\omega$ play the role of a translational gauge potential and a rotational gauge potential respectively. Throughout this description one can consider GR a pure gauge theory of gravity.

**The Tetrad Condition in MAG (ii)**

$$\underset{i}{\lambda}^{\nu}_{||\mu} = 0, \tag{15}$$

where $;^{\mu}$ is the absolute derivative of a non-metric quanity.

$$\Gamma^{\alpha}_{\mu\nu} = \underset{i}{\lambda}^{\alpha}(\underset{i}{\lambda}_{\mu,\nu} + \hat{\omega}^{\cdot j \cdot}_{i \cdot\cdot \nu} \underset{j}{\lambda}^{\alpha}). \tag{16}$$

The curvature tensor is defined to be

$$R^{i}_{j\gamma\delta} = \hat{\omega}^{i}_{j\gamma,\delta} - \hat{\omega}^{i}_{j,\delta\gamma} + \hat{\omega}^{k}_{j\gamma}\hat{\omega}^{i}_{k\delta} - \hat{\omega}^{k}_{j\delta}\hat{\omega}^{i}_{k\gamma}. \tag{17}$$

Thus, it is obvious to figure out $\underset{i}{\lambda}^{\mu}$ and $\hat{\omega}$ act in MAG theory as a translational gauge potential and a rotational gauge potential respectively. Provided that the associated curvature $R^{i}_{j\gamma\delta}$ in (17) may be expressed by its associated spin connection that gives rise to define matter. Accordingly, one can say without ambiguity that MAG theory is a gravitational theory invariant under gauge transformation.

## 2. Paths and Path Deviations in MAG: The Bazanski Approach

We suggest the following Lagrangian to obtain path and path deviation equations for MAG theory as an extension the the ususal Bazanski Lagrangian [8]:

$$L = g_{\mu\nu}U^{\mu}\frac{D\Psi^{\nu}}{Ds}, \tag{18}$$

where , $g_{\mu\nu}$ is the metric tensor, $U^{\mu}$, is a unit tangent vector of the path whose parameter is $s$, and $\Psi^{\nu}$ is the deviation vector associated to the path $(s)$, $\frac{D}{Ds}$ is the associated absolute derivative with respect to the parameter $s$ defind in the following way:

$$\frac{DA^{\alpha}}{Ds} \stackrel{\text{def}}{=} \frac{dU^{\alpha}}{ds} + \Gamma^{\alpha}_{\beta\delta}A\beta U^{\delta}, \tag{19}$$

in which $A^{\mu}$ is an arbitray vector.

Applying the Euler Lagrange equation, by taking the variation concerning the deviation tensor such that delta:

$$\frac{d}{ds}\frac{\partial L}{\partial \dot{\Psi}^{\mu}} - \frac{\partial L}{\partial \Psi^{\mu}} = 0, \tag{20}$$

to obtain the geodesic equation

$$\frac{DU^{\mu}}{Ds} = -g^{\mu\delta}\Omega_{\delta\rho\sigma}U^{\rho}U^{\sigma}, \tag{21}$$

and taking the variation with respect the the unit vector $U^{\mu}$

$$\frac{d}{ds}\frac{\partial L}{\partial U^{\mu}} - \frac{\partial L}{\partial x^{\mu}} = 0, \tag{22}$$

to obtain the geodesic deviation equation,

$$\frac{D^2\Psi^{\mu}}{Ds^2} = R^{\mu}_{\nu\rho\sigma}U^{\nu}U^{\rho}\Psi^{\sigma} - (g^{\mu\delta}\Omega\delta\rho\sigma U^{\rho}U^{\sigma})_{||\delta}\Psi^{\delta}. \tag{23}$$

It can be found that the term $g^{\mu\delta}\Omega\delta\rho\sigma U^{\rho}U^{\sigma}$ is acting as an external force due to the non-metricity condition.

### 2.1. On the Relation Between Spin Tensor and The Deviation Vector for MAG

In this part, we are going to extend the relationship obtained in [9] to derive the corresponding spin equations and their corresponding spin deviation equations.

Equations of spinning motion,the case of $P^{\alpha} = mV^{\alpha}$ can be related to path equation (21) if one follows the following transformation

$$\bar{V}^{\mu} = V^{\mu} + \beta\frac{D\Phi^{\mu}}{Ds}, \tag{24}$$

where $\bar{V}^{\alpha}$ is a unit tangent vector with respect to the parameter, such that $\bar{V}^{\alpha} = \frac{dx^{\mu}}{d\bar{s}+}$ , $\bar{s}$. By taking the covariant derivative on both sides one obtains:

$$\frac{D\bar{V}^{\alpha}}{D\bar{s}} = \frac{D}{Ds}(V^{\mu} + \beta\frac{D\Phi^{\mu}}{Ds})\frac{ds}{d\bar{s}}. \tag{25}$$

Substituting equations (19) and (21) in (25) to get

$$\frac{\nabla\bar{V}^{\alpha}}{\nabla\bar{s}} = (-g^{\alpha\beta}\Omega_{\beta\rho\delta}U^{\rho}U^{\delta} + \beta(R^{\alpha}_{\nu\rho\sigma}U^{\nu}U^{\rho}\Psi^{\sigma} - (g^{\mu\delta}\Omega\delta\rho\sigma U^{\rho}U^{\sigma})_{||\delta}\Psi^{\delta}))\frac{ds}{d\bar{s}}, \tag{26}$$

let us assume the following.

Taking $\beta = \frac{s}{m}$ [10], where $s$ the spin of the object and $m$ its mass

$$\bar{S}^{\mu\nu} = s(V^{\alpha}\Phi^{\beta} - V^{\beta}\Phi^{\alpha}). \tag{27}$$

Thus, we get

$$\frac{D\bar{V}^{\alpha}}{D\bar{s}} = (-g^{\alpha\beta}\Omega_{\beta\rho\delta}U^{\rho}U^{\delta} + \frac{1}{2m}R^{\alpha}_{\nu\rho\sigma}U^{\nu}S^{\rho\sigma} - \frac{s}{m}(g^{\mu\delta}\Omega_{\delta\rho\sigma}U^{\rho}U^{\sigma})_{||\delta}\Psi^{\delta}))\frac{ds}{d\bar{s}}\frac{ds}{d\bar{s}}. \tag{28}$$

The Papapetrou-like equation has an additive expression $g^{\alpha\beta}\Omega_{\beta\rho\delta}U^{\rho}U^{\delta} + \frac{s}{m}(g^{\mu\delta}\Omega_{\delta\rho\sigma}U^{\rho}U^{\sigma})_{||\delta}\Psi^{\delta}$ due to the effect of non-metricity tensor.

The above equation can be obtained by assuming the following Lagrangian;

$$L = U_{\nu}\frac{D\Psi^{\nu}}{+}\frac{1}{2m}(R_{\mu\nu\rho\sigma}U^{\nu}S^{\rho\sigma} - \frac{s}{m})\Omega_{\delta\rho\sigma}U^{\rho}U^{\sigma}_{||\delta})\Psi^{\delta})^{\mu}_{\Psi}. \tag{29}$$

If we take the variation with respect to $\Psi^{\alpha}$ we obtain

$$\frac{DU_{\alpha}}{Ds} = \frac{1}{2m}R_{\alpha\nu\rho\sigma}U^{\nu}S^{\rho\sigma}. \tag{30}$$

The contravarient form becomes:

$$\frac{Dg^{\mu\alpha}U_\alpha}{Ds} + \frac{g^{\mu\alpha}}{Ds}U_\alpha = \frac{1}{2m}R^{\mu}_{\nu\rho\sigma}U^\nu S^{\rho\sigma} + \frac{s}{m})(\Omega\delta\rho\sigma U^\rho U^\sigma)_{||\delta}\Psi^\delta, \tag{31}$$

$$\frac{DU^\alpha}{Ds} + g^{\alpha\beta}\Omega_{\beta\rho\delta}U^\rho U^\delta = \frac{1}{2m}R^{\mu}_{\nu\rho\sigma}U^\nu S^{\rho\sigma} - \frac{s}{m})\Omega\delta\rho\sigma U^\rho U^\sigma, \tag{32}$$

which can be reduced to equation (23).

## 3. On the relationship between Spinning Density Tensor and Spinning Tensor

The concept of spinning density tensor and its relationship with spinning tensor may be found in Weynssof Tensor [5] which is described as follows:

$$S^{\rho\mu\nu} = S^{\mu\nu}U^\rho, \tag{33}$$

where $S^{\rho\mu\nu}$ is the spin density tensor, $S^{\mu\nu}$ is the spin tensor, and $U^\rho$ is the four vector velocity. Differentiating both sides covariently, one obtains

$$\frac{DS^{\rho\mu\nu}}{Ds} = \frac{DS^{\mu\nu}}{Ds}U^\rho + \frac{DU^\rho}{Ds}S^{\mu\nu}. \tag{34}$$

Accordingly, equations of path equation and spin equation can be derived from the corresponding Bazanski lagrangian

$$L = g_{\mu\nu}U^\mu\frac{D\Psi^\nu}{Ds} + S_{\mu\nu}\frac{D\Psi^{\mu\nu}}{Ds}. \tag{35}$$

From the geodesic equation one obtains by taking the variation with respect to $\Psi$ to get the geodesic

$$\frac{DU^\mu}{Ds} = -g^{\mu\delta}(\Omega\delta\rho\sigma U^\rho U^\sigma, \tag{36}$$

and the spin equation may be obtained after taking the variation with respect to $\Psi^{\mu\nu}$

$$\frac{DS^{\mu\nu}}{Ds} = -g^{\mu\delta}g^{\nu\kappa}(\Omega_{\sigma\delta\alpha}g_{\kappa\beta} + \Omega_{\sigma\kappa\beta}g_{\nu\alpha})S^{\alpha\beta}U^\sigma. \tag{37}$$

Thus we can find out The following Lagrangian:

$$\frac{DS^{\rho\mu\nu}}{Ds} = -U^\rho g^{\mu\delta}g^{\nu\kappa}(\Omega_{\sigma\delta\alpha}g_{\kappa\beta} + \Omega_{\sigma\kappa\beta}g_{\nu\alpha})S^{\alpha\beta}U^\sigma - (g^{\rho\delta}(\Omega\delta\kappa\sigma U^\kappa U^\sigma)S^{\mu\nu}. \tag{38}$$

Such an equation is govered by means of implementing its corresponding Bazanski Lagrangian which is described as follows:

$$L = S_{\rho\mu\nu}\frac{D\Psi^{\rho\mu\nu}}{Ds}. \tag{39}$$

In which by taking the variation with respect $\Psi^{\alpha\beta\gamma}$ we obtain equation

$$\frac{DS^{\rho\mu\nu}}{Ds} = -g^{\mu\xi}g^{\nu\eta}g^{\rho\zeta}(Q_{\sigma\beta\gamma}g_{\alpha\xi} + Q_{\sigma\gamma\alpha}g_{\beta\eta} + Q_{\sigma\alpha\beta}g_{\gamma\zeta})S^{\alpha\beta\gamma}U^\sigma. \tag{40}$$

Now, if we express $S^{\rho\mu\nu} = U^\mu S^{\nu\rho}$ then (40) becomes:

$$\frac{DS^{\rho\mu\nu}}{Ds} = -g^{\mu\xi}g^{\nu\eta}g^{\rho\zeta}(Q_{\sigma\beta\gamma}g_{\alpha\xi} + Q_{\sigma\gamma\alpha}g_{\beta\eta} + Q_{\sigma\alpha\beta}g_{\gamma\zeta})U^\alpha S^{\beta\gamma}, \tag{41}$$

which reduces to

$$\frac{DS^{\rho\mu\nu}}{Ds} = -g^{\mu\xi}g^{\nu\eta}g^{\rho\zeta}(Q_{\sigma\beta\gamma}g_{\alpha\xi}U^\alpha S^{\beta\gamma} + Q_{\sigma\gamma\alpha}g_{\beta\eta}U^\alpha S^{\beta\gamma} + Q_{\sigma\alpha\beta}g_{\gamma\zeta}U^\alpha S^{\beta\gamma}), \tag{42}$$

$$\frac{DS^{\rho\mu\nu}}{Ds} = -g^{\mu\xi}g^{\nu\eta}g^{\rho\zeta}(Q_{\sigma\beta\gamma}g_{\alpha\xi}U^\alpha S^{\beta\gamma} + Q_{\sigma\gamma\alpha}g_{\beta\eta}U^\alpha S^{\beta\gamma} + Q_{\sigma\alpha\beta}g_{\gamma\zeta}U^\alpha S^{\beta\gamma}). \tag{43}$$

Using the following identity on both equations (42) and (43)

$$A^\delta_{;\nu\rho} - A^\delta_{;\rho\nu} = R^\delta_{\beta\nu\rho}A^\beta, \tag{44}$$

such that $A^{\delta}$ is an arbitrary vector,and the Riemannian curvature tensor $\bar{R}^{\sigma}_{\gamma\mu\nu}$ is defined as follows:

$$\bar{R}^{\rho}_{\gamma\mu\nu} = \left\{ \begin{matrix} \rho \\ \gamma\nu, \mu \end{matrix} \right\} - \left\{ \begin{matrix} \rho \\ \gamma\mu, \nu \end{matrix} \right\} + \left\{ \begin{matrix} \delta \\ \gamma\nu \end{matrix} \right\} \left\{ \begin{matrix} \rho \\ \gamma\delta, \mu \end{matrix} \right\} - \left\{ \begin{matrix} \delta \\ \gamma\mu \end{matrix} \right\} \left\{ \begin{matrix} \rho \\ \gamma\delta, \nu \end{matrix} \right\}. \tag{45}$$

Multiplying both sides with arbitrary vectors, $U^{\rho}\Psi^{\beta}$ as well as using the following condition

$$U^{\alpha}_{;\rho}\Psi^{\rho} = \Psi^{\alpha}_{;\rho}U^{\rho}, \tag{46}$$

and $\Psi^{\alpha}$ is its deviation vector associated to the vector tangent $U^{\alpha}$. Also in a similar way:

$$S^{\alpha\beta}_{||\rho}\Psi^{\rho} = \Psi^{\mu\nu}_{||\rho}U^{\rho}, \tag{47}$$

in which can be generalized in the following sense

$$S^{\rho\alpha\beta}_{||\delta}\Psi^{\delta} = \Psi^{\rho\mu\nu}_{||\delta}U^{\delta}. \tag{48}$$

Thus, we obtain the corresponding deviation equation:

$$\frac{D^2\Psi^{\rho\mu\nu}}{Ds^2} = S^{\delta[\mu\nu}\bar{R}^{\rho]}_{\delta\alpha\beta}U^{\alpha}\Psi^{\beta} - (g^{\mu\xi}g^{\nu\eta}g^{\rho\zeta}Q_{\sigma\beta\gamma}g_{\alpha\xi}U^{\alpha}S^{\beta\gamma} + Q_{\sigma\gamma\alpha}g_{\beta\eta}U^{\alpha}S^{\beta\gamma} + Q_{\sigma\alpha\beta}g_{\gamma\zeta}U^{\alpha}S^{\beta\gamma})_{||\kappa}\Psi^{\kappa}. \tag{49}$$

### 3.1. Spinning Tensor and Spinning Desnity Tensor Equations : The Papapetrou- Like Equations

If the spin density is related to momentum vector rather than four vector velocity.

$$\bar{S}^{\rho\mu\nu} = S^{\mu\nu}P^{\rho}, \tag{50}$$

where $P^{\rho}$ is the momentum in which it is relating to $S^{\rho\mu\nu}$ in the following sense

$$\bar{S}^{\rho\mu\nu} = S^{\mu\nu}(mU^{\rho} + U_{\delta}\frac{DS^{\rho\delta}}{Ds}), \tag{51}$$

i.e.

$$\bar{S}^{\rho\mu\nu} = S^{\mu\nu}(mU^{\rho} + U_{\delta}(P^{\rho}U^{\delta} - P^{\delta}U^{\rho})), \tag{52}$$

such that

$$P^{\mu} = mU^{\mu} + U_{\nu}\frac{DS^{\mu\nu}}{Ds}. \tag{53}$$

Differentiating both sides covariently of (52) to get

$$\frac{D\bar{S}^{\rho\mu\nu}}{Ds} = \frac{DS^{\mu\nu}}{Ds}P^{\rho} + \frac{DP^{\rho}}{Ds}S^{\mu\nu}. \tag{54}$$

We suggest the equivalent Bazanski Lagrangian

$$L = \bar{S}_{\rho\mu\nu}\frac{D\bar{\Psi}^{\rho\mu\nu}}{Ds} + f_{\rho\mu\nu}\bar{\Psi}^{\rho\mu\nu}. \tag{55}$$

To become after taking the variation with respect to its corresponding deviation tensor $\Psi^{\rho\mu\nu}$

$$\frac{D\bar{S}^{\rho\mu\nu}}{Ds} = f^{\rho\mu\nu}, \tag{56}$$

in which

$$f^{\rho\mu\nu} = \frac{DS^{\mu\nu}}{Ds}P^{\rho} + \frac{DP^{\rho}}{Ds}S^{\mu\nu}, \tag{57}$$

to be

$$f^{\rho\mu\nu} = (P^{\mu}U^{\nu} - P^{\nu}U^{\mu})P^{\rho} + \frac{1}{2}R^{\rho}_{\delta\mu\nu}S^{\mu\nu}U^{\delta}S^{\mu\nu}, \tag{58}$$

where

$$f_{mu} = \frac{1}{2} R^{\mu}_{\nu\rho\delta} S^{\rho\delta} U^{\nu} \tag{59}$$

is regarded as a spin force,and $R^{\alpha}_{\beta\rho\sigma}$ is the Riemann curvature, and

$$M^{\mu\nu} = P^{\mu} U^{\nu} - P^{\nu} U^{\mu} \tag{60}$$

such that $U^{\alpha}$ is the unit tangent vector. In a similar way as performed in [5], by taking the variation with respect to $\Psi^{\mu}$ and$\Psi^{\mu\nu}$ simultaneously one obtains

$$\frac{D\bar{P}^{\mu}}{D\bar{s}} = f^{\mu} - g^{\mu\delta} Q_{\sigma\delta\lambda} P^{\lambda} U^{\delta} U^{\sigma}, \tag{61}$$

and

$$\frac{D\bar{S}^{\mu\nu}}{D\bar{s}} = M^{\mu\nu} - g^{\mu\alpha} g^{\nu\beta} (Q_{\sigma\beta\nu} g_{\alpha\xi} + Q_{\sigma\alpha\mu} g_{\nu\eta}) S^{\xi\eta} U^{\sigma}. \tag{62}$$

### 3.2. Spinning and Spinning Deviation Equations with Precession

It is well known that equation of spinning charged objects in the presence of gravitational field have been studied extensively [4]. This led us to suggest its corresponding Lagrangian formalism, using a modified Bazanski Lagrangian [2], for a spinning and precessing object and their corresponding deviation equation in Riemanian geometry in the following way

$$L = g_{\alpha\beta} P^{\alpha} \frac{D\Psi^{\beta}}{Ds} + S_{\alpha\beta} \frac{D\Psi^{\alpha\beta}}{Ds} + F_{\alpha} \Psi^{\alpha} + M_{\alpha\beta} \Psi^{\alpha\beta}, \tag{63}$$

where

$$P^{\alpha} = mU^{\alpha} + U_{\beta} \frac{DS^{\alpha\beta}}{DS}. \tag{64}$$

Taking the variation with respect to $\Psi^{\mu}$ and $\Psi^{\mu\nu}$ simultaneously we obtain

$$\frac{DP^{\mu}}{DS} = F^{\mu} - g^{\mu\rho} Q_{\sigma\rho\delta} P^{\delta} U^{\sigma} - g^{\mu\delta} Q_{\sigma\delta\rho} U^{\rho} U^{\sigma} \tag{65}$$

and

$$\frac{DS^{\mu\nu}}{DS} = M^{\mu\nu} - g^{\mu\xi} g^{\nu\eta} (Q_{\sigma\eta\beta} g_{\xi\alpha} + Q_{\sigma\xi\alpha} g_{\eta\beta}) S^{\alpha\beta} U^{\sigma}, \tag{66}$$

where $P^{\mu}$ is the momentum vector,

$$F^{\mu} = \frac{1}{2} R^{\mu}_{\nu\rho\delta} S^{\rho\delta} U^{\nu}, \tag{67}$$

and $R^{\alpha}_{\beta\rho\sigma}$ is the Riemann curvature, $\frac{D}{Ds}$ is the covariant derivative with respect to a parameter $S$,$S^{\alpha\beta}$ is the spin tensor,

$$M^{\mu\nu} = P^{\mu} U^{\nu} - P^{\nu} U^{\mu}, \tag{68}$$

regarding $U^{\mu}$ is the unit tangent vector to the geodesic.

Using the following identity on both equations (4.59) and (4.60)

$$A^{\mu}_{;\nu\rho} - A^{\mu}_{;\rho\nu} = R^{\mu}_{\beta\nu\rho} A^{\beta}, \tag{69}$$

where $A^{\mu}$ is an arbitrary vector, and multiplying both sides with arbitrary vectors, $U^{\rho}\Psi^{\nu}$ as well as using the following condition [5]

$$U^{\alpha}_{||\rho} \Psi^{\rho} = \Psi^{\alpha}_{||\rho} U^{\rho}, \tag{70}$$

and $\Psi^{\alpha}$ is its deviation vector associated to the unit vector tangent $U^{\alpha}$. Also in a similar way:

$$S^{\alpha\beta}_{||\rho} \Psi^{\rho} = \Phi^{\alpha\beta}_{||\rho} U^{\rho}. \tag{71}$$

One obtains the corresponding deviation equations

$$\frac{D^2\Psi^{\mu}}{DS^2} = R^{\mu}_{\nu\rho\sigma} P^{\nu} U^{\rho} \Psi^{\sigma} + (F^{\mu} - g^{\mu\rho} Q_{\sigma\rho\delta} P^{\delta} U^{\sigma})_{||\kappa} \Psi^{\kappa} \tag{72}$$

and

$$\frac{D^2\Psi^{\mu\nu}}{DS^2} = S^{\rho[\mu} R^{\nu]}_{\rho\sigma\epsilon} U^{\sigma} \Psi^{\epsilon} + (M^{\mu\nu} g^{\mu\xi} g^{\nu\eta} (Q_{\sigma\eta\beta} g_{\xi\alpha} + Q_{\sigma\xi\alpha} g_{\eta\beta}) S^{\alpha\beta} U^{\sigma})_{||\kappa} \Psi^{\kappa}. \tag{73}$$

## 4. Discussion and Concluding Remarks

In the present work, we obtained the equations of spinning object with and without precession and obtained a translational step from the path equation to spinning equation using the relationship (3.21) We also related the spin tensor with path and path deviation as shown in (4.46) . This form can be useful to emphasize its exactness when we apply its associated Bazanski Lagrangian as to obtain the required spinning motion and its deviation tensor. Such an application can be regarded in the sense of describing theory of orbits in the framework of alternative theories of gravity. In otherwords, if we have a test particle transformed moving on a geodesic and translated by its deviation vector, the resultant new path can categorize as the behavior of spinning satisfying the usual Papaetrou equation [10] using a special type of parameter transformations. This type of study will be examined in our future work.

Moreover, the arising features that is appeared due to the non-metricity is appeared in euations(4.63) and (4.64) is counted on the path equation and its corresponding path deviation ones whether for the momentum the four vector velocity or the spinning tensor. Such an effect is increasing when we deal with equation of motion for different degrees of tensors i.e. $U^{\alpha}$ has a non-metricity quantity $g^{\alpha\xi}Q_{\sigma\mu\nu}g_{\delta\xi}U^{\sigma}U^{\delta}U^{\nu}$ , $S^{\mu\nu}$ has its corresponding non-metricity $g^{\mu\xi}g^{\nu\eta}(Q_{\sigma\eta\beta}g_{\xi\alpha}+Q_{\sigma\xi\alpha}g_{\eta\beta})S^{\alpha\beta}U^{\sigma}$ and for $S^{\rho\mu\nu}$ we have $g^{\mu\xi}g^{\nu\eta}g^{\rho\zeta}(Q_{\sigma\beta\gamma}g_{\alpha\xi}U^{\alpha}S^{\beta\gamma}+Q_{\sigma\gamma\alpha}g_{\beta\eta}U^{\alpha}S^{\beta\gamma}+Q_{\sigma\alpha\beta}g_{\gamma\zeta}U^{\alpha}S^{\beta\gamma})$ . From this perspective we can figure out that the effect of non-metricity is incresed due to the degree of the tensor equation. The nature of the nn-metricity tensor $Q_{\sigma\mu\nu}$ could be decomposed into different quantities that may lead to interactions with different fields which will be discussed in our forthcoming paper.

Eventually, it is worth mentioning that one of very crucial applications of path and path deviation equations is the problem of stability with respect to the non-metricity conditon for the Spinning Deviation Tensors of MAG may give rise to a relationship with the deviation vector of space-time. In other words, the solution of deviation equations for a test particle or spinning object can determine the system's behavior. To be more precise, if the deviation vector has led to a periodic function this means that the object may perturb around its original path which leads. But if the solution of gives styles of divergence functions, the object is no longer having a stable path. the problem of stability is connected with the type of the obtained gravitational field and the deviation vector. This type of study is going to be discussed in our future work.

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