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ПРИМЕНЕНИЕ ОПТИМАЛЬНОГО УПРАВЛЕНИЯ ДЛЯ ПОСТРОЕНИЯ КОСМОЛОГИЧЕСКИХ МОДЕЛЕЙ В ГЕОМЕТРИИ ЛИРЫ

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Рассмотрена задача построения космологических моделей в геометрии Лиры. Предложено использование принципов оптимального управления для построения замкнутой космологической модели. Сформулирована задача оптимального управления, где подлежащей максимизации целевой функцией является время до прекращения сжатия. Для поставленной задачи получены условия оптимальности и частное решение.

Ключевые слова: космологические модели, уравнения Эйнштейна, многообразие Лиры, оптимальное управление.

AN OPTIMAL CONTROL APPROACH FOR COSMOLOGICAL MODELS IN LYRA'S GEOMETRY

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The problem of constructing cosmological models within the framework of Lyra's geometry is considered. An optimal control approach is proposed for constructing a closed cosmological model. An optimal control problem is formulated, whereby the objective function, to be maximized, is the time to stop contraction. Optimality conditions are derived and a partial solution is obtained.

Keywords: cosmological models, Einstein's equations, Lyra's manifold, optimal control.

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Introduction

In 1918, Weyl [1] proposed a more comprehensive theory extending Einstein's theory, integrating gravitation and electromagnetism into geometric descriptions. Following this, Lyra [2] introduced a modification to Riemannian geometry by incorporating a gauge function into the structure-less manifold,

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resulting in the natural emergence of the cosmological constant from the geometry. Subsequently, Sen [3] and Sen and Dunn [4] explored static solutions with finite density in Lyra's modified Riemannian geometry, akin to the static Einstein model. Many researchers have further investigated cosmological models rooted in Lyra's geometry [5–13].

The Friedmann-Robertson-Walker (FRW) models of the Universe are characterized by the metric

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right]. \tag{1}$$

Here, $a(t)$ represents the scale factor, and k can take values $-1, 0$, or 1, corresponding to open, flat, and closed Universes respectively. The variables r, θ and φ denote spherical polar coordinates, and t represents cosmic time. The field equations (with $c = G = 1$) based on Lyra's geometry, as obtained by Sen [3], in the normal gauge are given by

$$R_{ij} - \frac{1}{2}g_{ij}R + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}g_{ij}\phi_k\phi^k = -8\pi T_{ij}, \tag{2}$$

where g_{ij} and R_{ij} are, respectively, the metric and Ricci tensors, $\phi_j = (0, 0, 0, \beta(t))$ is the time-like displacement vector. T_{ij} is the energy momentum tensor which, assuming a perfect fluid, is given by

$$T_{ij} = -pg_{ij} + (p+\rho)U_i\, U_j, \tag{3}$$

where p and ρ are, respectively, the fluid pressure and density, and $U_i = (0, 0, 0, 1)$ is the fluid velocity vector.

For the FRW metric, the field equations reduce to

$$3H^2 = \frac{3\beta^2}{4} - \frac{3k}{a^2} + 8\pi\rho, \tag{4}$$

$$2\dot{H} = -3H^2 - \frac{3\beta^2}{4} - \frac{k}{a^2} - 8\pi p, \tag{5}$$

where an overdot denotes a derivative with respect to time t, and $H(t)$ is the Hubble parameter defined by

$$H = \frac{\dot{a}}{a}. \tag{6}$$

The equation of state is given by

$$p = \alpha\rho, \qquad \alpha = \text{Const}, \qquad 0 \le \alpha < 1. \tag{7}$$

Taking the time derivative of Eqs.(4),(5) and using Eqs.(6),(7), we obtain

$$6H\dot{H} = \frac{3}{2}\beta\dot{\beta} + \frac{6k}{a^2}H + 8\pi\dot{\rho}, \tag{8}$$

$$2\ddot{H} + 6H\dot{H} = \frac{-3}{2}\beta\dot{\beta} + \frac{2k}{a^2}H - 8\pi\alpha\dot{\rho}. \tag{9}$$

Several authors have made specific assumptions about $\rho(t)$ or $a(t)$ or $\beta(t)$ to obtain solutions for Einstein's modified field equations for the Friedmann metric within the context of Lyra's geometry. For instance, Beesham [14] assumed that the energy density of the universe is always equal to its critical value ($\rho = 3\dot{a}^2/a^2$) and consequently $\beta^2 = 4k/a^2$; Pradhan, Shahi, and Singh [15] obtained solutions using the deceleration parameter and considered three cases: $a(t) = k_2 e^{k_1 t}$, $a(t) = \alpha t^2 + \alpha_2 t + \alpha_3$ and $a(t) = M\sin(Bt) + N\cos(Bt) + B_1$ (where $k_1, k_2, \alpha, \alpha_2, \alpha_3, M, N, B$ and B_1 are constants); Singh [16] assumed that $\beta^2 = \lambda H^2$ (where λ is a positive constant); Darabi [17] assumed that $\beta(t)$ and $\rho(t)$ are independent quantities with no interaction, yielding $\beta = \beta_0(a_0/a)^3$, (where a_0 and β_0 are the scale factor and the displacement vector field corresponding to the Einstein static universe). Desikan [12] made the assumption $\beta^2(t) = \beta_0 H^2$ (where β_0 is a constant) and determined the conditions that govern whether a matter-dominated, flat universe undergoes decelerated or accelerated expansions.

In this study, instead of relying on arbitrary assumptions, we suggest an optimal control approach for constructing cosmological models. The literature on optimal control is extensive [18–21]. Some applications of optimal control to relativistic astrophysics and cosmology are reviewed in [22] and [23] . For a brief outline of optimal control problems see [24].

The structure of this paper is outlined as follows. In Section 1, we formulate an optimal control problem for a closed cosmological model. In Section 2 we present the optimality conditions. In Section 3, we derive a partial solution and describe the work in progress to obtain a complete solution. The paper ends with a concise conclusion.

1. An Optimal Control Problem

We consider a closed cosmological model, $k = 1$, where the universe expands then contracts. It is interesting to find a universe with maximum time T to stop contraction, namely to reach $\dot{a} = 0$.

For $k = 1$, Eqs. (8) and (9) lead to

$$2\ddot{H} + 6H\dot{H}(1+\alpha) = -\frac{3}{2}\beta\dot{\beta}(1-\alpha) + \frac{2}{a^2}H(1+3\alpha), \tag{10}$$

which can be put in the form

$$\dot{\beta} = \frac{-2}{3\beta(1-\alpha)}\left[2\ddot{H} + 6H\dot{H}(1+\alpha) - \frac{2}{a^2}H(1+3\alpha)\right]. \tag{11}$$

Introducing new variables

$$w := \dot{H}, \qquad u := \dot{w} = \ddot{H}, \qquad -1 \le u \le 1, \tag{12}$$

we take $\beta(t)$, $a(t)$, $H(t)$ and $w(t)$ as state functions, and $u(t)$ as the control function. Then the optimal control problem is to determine the control function $u(t)$ that maximizes the objective function T, namely

$$\max_{-1\le u\le 1} T = \int_0^T dt, \tag{13}$$

subject to the constraints

$$\dot{\beta} = \frac{-2}{3\beta(1-\alpha)}\left[2u + 6Hw(1+\alpha) - \frac{2}{a^2}H(1+3\alpha)\right], \qquad \beta(0) = \beta_0, \tag{14}$$

$$\dot{a} = aH, \qquad a(0) = a_0, \tag{15}$$

$$\dot{H} = w, \qquad H(T) = 0, \tag{16}$$

$$\dot{w} = u, \qquad w(0) = w_0, \tag{17}$$

where β_0, a_0 and w_0 are constants.

2. Optimality Conditions

The Hamiltonian for the system above is given by

$$S = 1 + \frac{-2}{3\beta(1-\alpha)}\left[2u + 6Hw(1+\alpha) - \frac{2}{a^2}H(1+3\alpha)\right]\lambda_1 + \lambda_2 aH + \lambda_3 w + \lambda_4 u, \tag{18}$$

where $\lambda_i, i = 1-4$, are the co-state functions corresponding, respectively, to the state functions $\beta(t)$, $a(t)$, $H(t)$ and $w(t)$.

According to Pontryagin's maximum principle, u should maximize S. Besides, the co-state functions should satisfy the adjoint equations

$$\dot{\lambda}_1 = -\frac{\partial S}{\partial \beta} = \frac{-2}{3\beta^2(1-\alpha)}\left[2u + 6Hw(1+\alpha) - \frac{2}{a^2}H(1+3\alpha)\right]\lambda_1, \quad \lambda_1(T) = 0, \tag{19}$$

$$\dot{\lambda}_2 = -\frac{\partial S}{\partial a} = \frac{8(1+3\alpha)}{3\beta(1-\alpha)a^3}H\lambda_1 - H\lambda_2, \qquad \lambda_2(T) = 0, \tag{20}$$

$$\dot{\lambda}_3 = -\frac{\partial S}{\partial H} = \frac{-2}{3\beta(1-\alpha)}\left[\frac{2}{a^2}(1+3\alpha) - 6(1+\alpha)w\right]\lambda_1 - a\lambda_2, \quad \lambda_3(0) = 0, \tag{21}$$

$$\dot{\lambda}_4 = -\frac{\partial S}{\partial w} = \frac{4(1+\alpha)}{\beta(1-\alpha)}H\lambda_1 - \lambda_3. \qquad \lambda_4(T) = 0. \tag{22}$$

Since the final time is not fixed (free), we have $S(T) = 0$, and thus

$$1 + w(T)\lambda_3(T) = 0. \tag{23}$$

3. Towards The Optimal Solution

We observe that the Hamiltonian $S(t)$ is linear in $u(t)$, so its maximum occurs at either boundary of u depending on the sign of the factor

$$m = \frac{-4}{3\beta(1-\alpha)}\lambda_1 + \lambda_4, \tag{24}$$

Therefore, S is maximized with respect to u by taking $u = +1$ for $m < 0$ and $u = -1$ for $m > 0$. When m changes sign, u switches to the other boundary, and this optimal control is termed bang-bang control [21]. The number of switches between boundaries will depend on the number of sign changes in m.

With $u = \pm 1$, Eq. (17) integrates to $w = \pm t + w_0$. Then, we find

$$H(t) = \pm\frac{1}{2}t^2 + w_0 t + H_0, \tag{25}$$

where H_0 is a constant to be determined.

Then, we obtain

$$a(t) = a_0 \exp\left[\pm\frac{1}{6}t^3 + \frac{1}{2}w_0 t^2 + H_0 t\right]. \tag{26}$$

From Eqs. (4) and (5) with $k = 1$, we obtain

$$\beta^2 = \frac{-4}{3(1-\alpha)}\left(2\dot{H} + 3H^2(1+\alpha) + \frac{1}{a^2}(1+3\alpha)\right). \tag{27}$$

We can derive formulas for β^2 and ρ from Eqs. (27) and (4) respectively using Eqs. (25) and (26). We obtain

$$\begin{aligned}\beta^2 &= \frac{-8}{3(1-\alpha)}\Bigg\{w_0 \pm t + \frac{3}{8}\left(\pm t^2 + 2w_0 t + 2H_0\right)^2(1+\alpha) + \frac{(1+3\alpha)}{2a_0^2} \\ &\times \exp\left[\frac{-1}{3}\left(\pm t^3 + 3w_0 t^2 + 6H_0 t\right)\right]\Bigg\},\end{aligned} \tag{28}$$

$$\begin{aligned}\rho &= \frac{1}{24\pi a_0^2(1-\alpha)^2}\Bigg\{9(1-\alpha)^2 \exp\left[\frac{-1}{3}\left(\pm t^3 + 3w_0 t^2 + 6H_0 t\right)\right] \\ &+ \frac{9a_0^2(1-\alpha)^2}{4}(\pm t^2 + 2w_0 t + 2H_0)^2 \\ &- 16a_0^2\Bigg(\pm t + w_0 + \frac{3(1+\alpha)}{8}(\pm t^2 + 2w_0 t + 2H_0)^2 \\ &+ \frac{1}{2a_0^2}(1+3\alpha)\exp\left[\frac{-1}{3}\left(\pm t^3 + 3w_0 t^2 + 6H_0 t\right)\right]\Bigg)^2\Bigg\}.\end{aligned} \tag{29}$$

Work is in prgress to obtain a complete solution. We need to solve the adjoint equations and determine the integration constants. Then, the factor m determines the behaviour of the control u, which, in turn, completely determines the state functions. We can, then, obtain a formula for the maximum time.

Conclusion

We formulated a cosmological model within the framework of Lyra's modified Riemannian geometry, employing an optimal control approach. By applying Pontryagin's maximum principle, we derived a solution, allowing for the determination of the time-dependent displacement field $\beta(t)$ and the energy density $\rho(t)$ for a closed FRW model. This approach offers a framework that can be extended to explore other cosmological and astrophysical models, particularly in scenarios where the governing equations are underdetermined. The proposed approach provides a deeper understanding of cosmic dynamics under alternative geometric structures.

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