УДК 530.12, 531.51

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РАСШИРЕННЫЙ КЛАСС ТОЧНЫХ КОСМОЛОГИЧЕСКИХ РЕШЕНИЙ НА ОСНОВЕ ПРЕОБРАЗОВАНИЙ ДАРБУ

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Рассматриваются форм-инвариантные преобразования уравнений космологической динамики для инфляционных моделей со скалярным полем на основе гравитации Эйнштейна. На основе комбинации различных форм-инвариантных преобразований получены расширенные преобразования Дарбу. Рассматривается метод построения цепочек точных решений уравнений космологической динамики на основе преобразований данного вида.

Ключевые слова: космологические модели, скалярное поле, реликтовые гравитационные волны..

THE EXTENDED CLASS OF EXACT COSMOLOGICAL SOLUTIONS BASED ON THE DARBOUX TRANSFORMATIONS

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Form-invariant transformations of the equations of cosmological dynamics for inflationary models with a scalar field based on Einstein gravity are considered. Based on a combination of various form-invariant transformations, extended Darboux transformations are obtained. A method for constructing chains of exact solutions of the equations of cosmological dynamics based on this transformations is considered.

Keywords: cosmological models, scalar field, relic gravitational waves.

PACS: 04.50.-h
DOI: 10.17238/issn2226-8812.2025.2.103-117

Introduction

The theory of cosmological inflation, based on the assumption of the existence of an accelerated expansion, which precedes the hot universe stage in the Big Bang theory, is the most consistent approach to describing the evolution of the early universe [1–3]. To describe the inflationary stage of the early universe, various models are considered, based on both Einstein gravity and its various modifications [1–3]. In this paper inflationary models based on Einstein gravity with a single scalar field will be considered.

The dynamic equations of a scalar field for a spatially flat Friedmann-Robertson-Walker (FRW) space-time can be noted as [1, 3]

$$H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{1}$$

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2, \tag{2}$$

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi). \tag{3}$$

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In this system of dynamic equations $a(t)$ is the scale factor, $H(t) = \dot{a}(t)/a(t)$ is the Hubble parameter, where the Einstein gravitational constant $\kappa = 1$, $\phi(t)$ is the scalar field, and $V(\phi)$ is a potential of a scalar field. A dot denotes the derivative with respect to the cosmic time t, and a prime denotes the derivative with respect to the scalar field.

In the system (1)–(3) one has two independent equations only, and to construct the exact cosmological solution one can solve two dynamic equations (1)–(2) only. There are a different methods to generate the exact solutions of system (1)–(3) (for the review, see [3–5]).

One of such approach based on the following representation of the dynamic equations [3–5]

$$V(\phi(t)) = 3H^2 + \dot{H}, \tag{4}$$

$$\dot{H} = -\frac{1}{2}\dot{\phi}^2, \tag{5}$$

where the exact solutions can be generated by the choice of a scalar field evolution $\phi = \phi(t)$ or dynamic of the universe's expansion $H = H(t)$.

At present, cosmological models with different scalar field potentials $V(\phi)$ are considered, that determines a different ways of implementing the inflationary scenario [6]. Thus, the only true expression for the potential of a scalar field is unknown.

It should be noted that inflationary models must contain a combination of Friedmann solutions and (quasi) de Sitter solutions which present the basis of an actual description of the evolution of the universe. In the context of the inflationary paradigm, the early universe expands rapidly for some time and then it transits to a power-law expansion regime without acceleration corresponding to the Friedmann solution [1–3].

In this paper we consider the Schrödinger-like representation of the equation (5) and use the Darboux transformations to construct the chains of exact cosmological solutions corresponding to compound exponential power-law dynamic of the universe's expansion. Such solutions define the explicit dependencies of a scalar field, Hubble parameter and the potential from cosmic time that implies the reconstructing the shape of an potential of scalar field.

1. Schrödinger-like representation of cosmological dynamic equations

The Schrödinger representation of the first dynamic equation was proposed in [7–9] The basis of this approach is representation of equation (4) as the Schrödinger-like one in terms of cosmic time

$$\ddot{\Psi} - U(t)\Psi = 0, \tag{6}$$

with corresponding second dynamic equation

$$\dot{\phi}^2 = -\frac{2}{3}\frac{d^2}{dt^2}\ln\Psi, \tag{7}$$

where $U(t) = 3V(t)$ and $\Psi \equiv Ca^3(t)$ which can be considered as some partial solution of equation

$$\ddot{f} - U(t)f = 0, \tag{8}$$

with the same potential, and C is an arbitrary constant.

Further, one can use the Darboux transformations

$$\tilde{U} = U - 2\left\{\frac{d^2}{dt^2}\ln f(t)\right\}, \tag{9}$$

$$\tilde{\Psi} = \dot{\Psi} - \Psi\left\{\frac{d}{dt}\ln f(t)\right\}, \tag{10}$$

where function $f(t)$ can be defined as

$$f(t) = \Psi(t)\left(C_1 + C_2\int[\Psi(t)]^{-2}\,dt\right), \tag{11}$$

to construct the exact cosmological solutions of new Schrödinger-like equation

$$\ddot{\tilde{\Psi}} - \tilde{U}(t)\tilde{\Psi} = 0, \tag{12}$$

on the basis of some known ones.

The disadvantage of this approach is that often it is impossible to find the field evolution $\phi = \phi(t)$ in explicit form equation (7) after applying Darboux transformations. Thus, the shape of the scalar field potential $V(\phi)$ for such solutions remains unclear.

The other approach based on the representation of equation (4) as Schrödinger-like one in terms of a scalar field is considered in [10], and dynamic equations were represented as

$$\left[-\frac{d^2}{d\phi^2} + U(\phi)\right]\psi(\phi) = 0, \tag{13}$$

$$V'_\phi = 6\left[1 - \frac{2}{3}U(\phi)\right]\psi\psi'_\phi, \tag{14}$$

$$\dot{\phi} = -2\psi'_\phi, \tag{15}$$

where $\psi(\phi) \equiv H(\phi)$.

The Darboux transformations (9)–(12) give the new potential $\tilde{V}$ and the Hubble parameter $\tilde{H}$, however, solutions of equation (15) in explicit form for the new Hubble parameter $\tilde{H}$ can be obtained for a limited number of models only.

The Schrödinger-like representation of the second dynamic equation (5) is considered in [11] as

$$\frac{d^2a(\tau)}{d\tau^2} + \frac{1}{2}\left[\frac{d\phi(\tau)}{d\tau}\right]^2 a(\tau) = 0, \tag{16}$$

$$\frac{d}{dt} \equiv a\frac{d}{d\tau}. \tag{17}$$

In this case, the Darboux transformations (9)–(12) give a new scale factor $\tilde{a}$ and kinetic energy $\frac{1}{2}\left[\frac{d\tilde{\phi}(\tau)}{d\tau}\right]^2$ only but not the transformation of the field itself. Thus, in this case, the problem again reduces to finding solutions to equation (5) in terms of a new time parameter τ.

Therefore, the use of Darboux transformations (9)–(12) for these methods for reducing the equations of cosmological dynamics to the Schrödinger-like equation seems to be an ineffective tool to find the evolution of a scalar field $\phi = \phi(t)$ in explicit form and the shape of its potential $V(\phi)$ as well.

Thus, to solve the problem of constructing the exact cosmological solutions for a complicated dynamics of the expansion of the universe, a new method is needed that will allow one to use the Darboux transformations and combine them with other form-invariant transformations as well.

2. Form-invariant transformations of the second dynamic equation

Form-invariant transformations (FIT) associated with various types of symmetries preserve the form of dynamic equations, and application of FIT to equations (4)–(3) can be written as

$$FIT : \{\phi, H, V\} \to \{\tilde{\phi}, \tilde{H}, \tilde{V}\}, \tag{18}$$

where $\{\phi, H, V\}$ are initial exact solutions of these equations and $\{\tilde{\phi}, \tilde{H}, \tilde{V}\}$ are new exact solutions.

To generate exact solutions, we consider form-invariant transformations of the second dynamic equation

$$\dot{\phi}^2 = -2\dot{H}, \tag{19}$$

i.e. we will consider the transformations $\{\phi, H\} \to \{\tilde{\phi}, \tilde{H}\}$ to obtain the solutions of equation

$$\dot{\tilde{\phi}}^2 = -2\dot{\tilde{H}}. \tag{20}$$

In general case, the relations between solutions $\{\phi, H\}$ and $\{\tilde{\phi}, \tilde{H}\}$ can be defined as follow

$$\tilde{\phi}(t) = \phi(t) + \theta(t), \tag{21}$$

$$\tilde{H}(t) = H(t) + \eta(t), \tag{22}$$

where $\theta(t)$ and $\eta(t)$ are some function of cosmic time connected by equation

$$\dot{\eta} + \left(\frac{1}{2}\dot{\theta} + \dot{\phi}\right)\dot{\theta} = 0, \tag{23}$$

and form-invariant transformations of equation (19) define this connection in explicit form.

Such transformations can be applied an arbitrary number of times, which determines the chains of exact solutions of equation

$$\dot{\phi}_i^2 = -2\dot{H}_i, \tag{24}$$

where i denotes the order of the transformations.

The potential of a scalar field as a function of cosmic time can be obtained from expression

$$V_i(t) = 3H_i^2 + \dot{H}_i, \tag{25}$$

for each i–solutions of equation (24).

The potential as a function of scalar field is defined in parametrical form by following way

$$V_i(\phi_i) = \begin{cases} V_i = V_i(t), \\ \phi_i = \phi_i(t). \end{cases}$$

The other way to obtain the expression for potential $V_i(\phi_i)$ is to find the dependence $H_i = H_i(\phi_i)$ from $H_i = H_i(t)$ and $t_i = t_i(\phi_i)$.

Further, one can use expression

$$V_i(\phi_i) = 3H_i^2(\phi_i) - 2\left(\frac{dH_i(\phi_i)}{d\phi_i}\right)^2, \tag{26}$$

based on the Ivanov-Salopec-Bond approach [3] to obtain the potential of a scalar field in explicit form.

In the other cases, for solutions without the explicit inverse dependence $t_i = t_i(\phi_i)$, it is possible to reconstruct the shape of the potential $V_i(\phi_i)$ from expressions $V_i = V_i(t)$ and $\phi_i = \phi_i(t)$.

Now, we consider some types of form-invariant transformations, which can be used to construct new exact cosmological solutions from known ones.

2.1. The shifts and dilations of a scalar field and Hubble parameter

A simple transformations of equation (19) are the combinations of the shifts and dilations of the scalar field ϕ and the Hubble parameter H.

Shifts transformations can be defined as

$$\phi \to \phi + c, \qquad H \to H + \lambda. \tag{27}$$

Therefore, one can obtain the field evolution $\phi(t)$ and Hubble parameter $H(t)$ from equation (19) up to the choice of constants c and λ that corresponds to the following transformation of a scale factor

$$a(t) \to a(t)\exp(\lambda t). \tag{28}$$

This result demonstrates that the simplest case of inflation for $\phi = 0$ is a pure exponential expansion with $a(t) \propto \exp(\lambda t)$ which induced by cosmological constant $\Lambda = V = 3\lambda_2^2$.

The other form-invariant transformations (combination of shifts and dilations) can be written as [3, 5]

$$\phi \to \gamma\sqrt{n}\phi + c, \qquad H \to nH + \lambda, \tag{29}$$

where $\gamma = \pm 1$, n and λ are an arbitrary constants.

The scale factor under these transformations changes as

$$a(t) \to [a(t)]^n \exp(\lambda t), \tag{30}$$

that corresponds to generalized exponential power-law dynamics.

Also, the choice of the sign of the constant n allows one use the transition between exact solutions for canonical and phantom scalar fields [3, 5].

As one can see, the repeated transformations (29) does not change the character of the dynamic, so their using to construct the exact cosmological solutions is limited by the law of expansion (30). Thus, to generate the exact cosmological solutions with different model's parameters $\{\phi, H, V\}$ one must use the other types of a form-invariant transformations of equation (19).

3. The Darboux class of exact cosmological solutions

In order to find the other type of form-invariant transformations of equation (19) we consider the representation of this equation as following system

$$\ddot{\psi} - u(t)\psi = 0, \tag{31}$$

$$u(t) = \ddot{\phi} - 2\dot{H}, \tag{32}$$

$$\psi(t) = \mu \exp(\phi(t)), \tag{33}$$

where (31) is the Schrödinger-like equation and μ is an arbitrary constant.

Now, we consider two Schrödinger-type equations

$$\ddot{\psi} - u(t)\psi = 0, \tag{34}$$

and

$$\ddot{\tilde{\psi}} - \tilde{u}(t)\tilde{\psi} = 0, \tag{35}$$

which can be represented as the following systems

$$u(t) = \ddot{\phi} - 2\dot{H}, \tag{36}$$

$$\psi(t) = \mu_1 \exp(\phi(t)), \tag{37}$$

$$\dot{\phi}^2 = -2\dot{H}, \tag{38}$$

and

$$\tilde{u}(t) = \ddot{\tilde{\phi}} - 2\dot{\tilde{H}}, \tag{39}$$

$$\tilde{\psi}(t) = \mu_2 \exp(\tilde{\phi}(t)), \tag{40}$$

$$\dot{\tilde{\phi}}^2 = -2\dot{\tilde{H}}, \tag{41}$$

where ψ and $\tilde{\psi}$ are partial solutions of the equations (34) and (35) respectively.

The connection between these solutions can be obtained from the Darboux transformations

$$\tilde{u} = u - 2\frac{d^2}{dt^2} \ln f(t), \tag{42}$$

$$\tilde{\psi} = \dot{\psi} - \psi \left\{ \frac{d}{dt} \ln f(t) \right\}, \tag{43}$$

where $f(t)$ is the general solution of the equation (34)

$$\ddot{f} - u(t)f = 0. \tag{44}$$

Based on transformations (42)–(44) one can write the connection between the exact solutions of equations (38) and (41) in following form

$$\tilde{\phi}(t) = \phi(t) + \chi(t), \tag{45}$$

$$\tilde{H}(t) = H(t) + \frac{\dot{f}}{f} + \frac{1}{2}\dot{\chi}, \tag{46}$$

$$\chi(t) = \ln\left[\frac{\mu_1}{\mu_2}\left(\dot{\phi} - \frac{\dot{f}}{f}\right)\right], \tag{47}$$

$$\ddot{f} - \left(\ddot{\phi} - 2\dot{H}\right) f = 0. \tag{48}$$

On the basis of the expression (38) equation (48) can be represented as

$$\ddot{f} - \left(\ddot{\phi} + \dot{\phi}^2\right) f = 0. \tag{49}$$

General solution of equation (49) is

$$f(t) = \exp[\phi(t)]\left\{c_1 + c_2\int \exp[-2\phi(t)]dt\right\}, \tag{50}$$

where c_1 and c_2 are an arbitrary constants.

After substituting the solution (50) into expressions (45)–(48) and redefining the constants as $c_3 \equiv -\frac{\mu_1 c_2}{\mu_2}$ we obtain transformation from equation (38) to (41)

$$\dot{\phi}^2 = -2\dot{H}, \quad \Rightarrow \quad \dot{\tilde{\phi}}^2 = -2\dot{\tilde{H}}, \tag{51}$$

which is defined by following connections

$$\tilde{\phi}(t) = \phi(t) + \ln\left[\frac{c_3\dot{\sigma}}{c_1 + c_2\sigma}\right], \tag{52}$$

$$\bar{H}(t) = H(t) + \frac{c_2\dot{\sigma}}{2(c_1 + c_2\sigma)}, \tag{53}$$

$$\dot{\sigma} \equiv \exp[-2\phi(t)]. \tag{54}$$

Thus, these transformations define the Darboux class of exact cosmological solutions.

4. Extended Darboux class of exact cosmological solutions

Firstly, from the definition (54) we can write

$$\phi(t) = -\frac{1}{2}\ln\dot{\sigma}, \tag{55}$$

and represent expression (52) as

$$\tilde{\phi}(t) = -\frac{1}{2}\ln\dot{\sigma} + \frac{1}{2}\ln\left[\frac{c_3^2\dot{\sigma}^2}{(c_1 + c_2\sigma)^2}\right] = \frac{1}{2}\ln\left[\frac{c_3^2\dot{\sigma}}{(c_1 + c_2\sigma)^2}\right]. \tag{56}$$

Secondly, after redefinitions of functions and corresponding constants

$$\tilde{\phi}(t) \to 2\tilde{\phi}(t), \qquad \dot{\sigma} \to (\dot{\sigma})^{-q/2}, \qquad \{c_1, c_2, c_3\} \to \{A, B, C\}, \tag{57}$$

where q is an arbitrary constant, from (38) and (41) we obtain the following relations

$$\tilde{\phi}(t) = \ln\left[\frac{A\dot{\sigma}}{(C\sigma + B)^2}\right], \tag{58}$$

$$\tilde{H}(t) = q^2 H(t) + \frac{2C\dot{\sigma}}{C\sigma + B}, \tag{59}$$

$$\dot{\sigma} \equiv \exp\left[q\phi(t)\right], \tag{60}$$

where constant $A \neq 0$.

For the case $C = 0$, $q = 1$, $A = B^2$ one has $\tilde{\phi} = \phi$ and $\tilde{H} = H$, i.e. the initial solutions of equation (38). Inverse transformations towards (58)–(60) are written similarly to direct ones up to the choice of constants that can be verified by trivial calculations.

Finally, after combining transformations (58)–(60) with shifts and dilations (29) one can define the recurrence relations

$$\phi_{k+1}(t) = \gamma\sqrt{n_k}\,\ln\left[\frac{A_k\dot{\sigma}_k}{(C_k\sigma_k + B_k)^2}\right], \tag{61}$$

$$H_{k+1}(t) = n_k\left[q_k^2 H_k(t) + \frac{2C_k\dot{\sigma}_k}{C_k\sigma_k + B_k}\right] + \lambda_k, \tag{62}$$

$$\dot{\sigma}_k \equiv \exp\left[q_k\phi_k(t)\right], \tag{63}$$

which connect the exact solutions of the cosmological dynamic equations

$$V_k(t) = 3H_k^2 + \dot{H}_k, \tag{64}$$

$$\dot{\phi}_k^2 = -2\dot{H}_k, \tag{65}$$

and

$$V_{k+1}(t) = 3H_{k+1}^2 + \dot{H}_{k+1}, \tag{66}$$

$$\dot{\phi}_{k+1}^2 = -2\dot{H}_{k+1}, \tag{67}$$

where index $k = 0, 1, 2, 3, ...$ defines the order of the transformations (61)–(63).

4.1. Chaotic inflation with quadratic potential

The differences between solutions (21)–(22) can be defined by expressions (52)–(54) and (25) in following form

$$\Delta\phi = \tilde{\phi} - \phi = \theta(t) = \ln\left[\frac{c_3\dot{\sigma}}{c_1 + c_2\sigma}\right], \tag{68}$$

$$\Delta H = \tilde{H} - H = \eta(t) = \frac{c_2\dot{\sigma}}{2(c_1 + c_2\sigma)}, \tag{69}$$

$$\Delta V = \tilde{V} - V = 3\eta^2(t) + \dot{\eta} + 6\eta(t)H(t). \tag{70}$$

with following connection

$$\dot{\sigma} \equiv \exp[-2\phi(t)], \tag{71}$$

corresponding to fulfillment of the condition (23).

Also, from equations (68) and (69) one has following expression

$$\eta(t) = \frac{c_2}{2c_3}\exp(\theta(t)), \tag{72}$$

which characterize the connection of differences between initial and new parameters of cosmological models for the Darbox class of exact solutions.

In principle, it is possible to generate exact cosmological solutions by specifying the differences θ or η. As the example, we consider the case $\theta = 0$, and from equation (68) we obtain

$$\sigma(t) = \sigma_0(t) = \frac{1}{c_2}\exp\left[-\frac{c_2(c_4 - t)}{c_3}\right] - \frac{c_1}{c_2}, \tag{73}$$

where c_4 is the constant of integration.

From equations (69) one has

$$\tilde{H}(t) = H(t) + \frac{c_2}{2c_3}, \tag{74}$$

and from (71) we obtain the linear evolution of a scalar field

$$\tilde{\phi}(t) = \phi(t) = \phi_0(t) = \alpha t + \beta, \tag{75}$$

where we redefine the constants as α and β, and the index 0 means that we will consider this evolution of the scalar field as the first in the chain of exact solutions.

For this law of the evolution of scalar field (75) from equations (19) and (25) (for $k = 0$) one has following exact solutions

$$H_0(t) = -\frac{\alpha^2}{2}t + \mu_0, \tag{76}$$

$$a_0(t) = Q_0 \exp\left[-\frac{t}{4}\left(\alpha^2 t - 4\mu_0\right)\right], \tag{77}$$

$$V_0(t) = 3\left(\frac{\alpha^2}{2}t + \mu_0\right)^2 - \frac{\alpha^2}{2}, \tag{78}$$

$$V_0(\phi_0) = 3\left[\frac{\alpha}{2}(\beta - \phi_0) + \mu_0\right]^2 - \frac{\alpha^2}{2}, \tag{79}$$

where μ_0 and Q_0 are the constants of integration.

These solutions correspond to chaotic inflation with quadratic potential and massive scalar field [13]. Also, solutions (76)–(79) was considered in [3].

Thus, applying transformations (68)–(71) k-times, one can construct a chain of exact cosmological solutions.

5. Degenerate form-invariant transformations

After redefinition of function $\sigma(t)$ and combining the result of the Darboux transformations (52)–(54) with shifts and dilations (27), (29) one can define the relations for extended Darboux class of exact cosmological solutions as

$$\phi_k(t) = \frac{1}{q_k}\ln\dot{\sigma}_k, \tag{80}$$

$$\dot{H}_k = -\frac{1}{2q_k^2}\left(\frac{\ddot{\sigma}_k}{\dot{\sigma}_k}\right)^2, \tag{81}$$

$$\phi_{k+1}(t) = \gamma\sqrt{n_k}\,\ln\left[\frac{A_k\dot{\sigma}_k}{(C_k\sigma_k + B_k)^2}\right], \tag{82}$$

$$H_{k+1}(t) = n_k\left[q_k^2 H_k(t) + \frac{2C_k\dot{\sigma}_k}{C_k\sigma_k + B_k}\right] + \lambda_k. \tag{83}$$

Thus, the exact solutions $\{\phi_k, H_k\}$ and $\{\phi_{k+1}, H_{k+1}\}$ of equation (19) are connected by relations (80)–(83) that corresponds to a chain of exact cosmological solutions induced by some initial solutions $\{\phi_0, H_0\}$ for $k = 0$.

Also, from equation (81) and the definition of Hubble parameter $H = \dot{a}/a$ one can obtain the transformation of a scale factor

$$a_{k+1}(t) = Q_{k+1}\left[a_k(t)\right]^{n_k q_k^2}\left[C_k\sigma_k(t) + B_k\right]^{2n_k}\exp\left(\lambda_k t\right), \tag{84}$$

where Q_{k+1} is the constant of integration.

Thus, each application of transformations (80)–(82) induces a cosmological model with a new type of dynamics of expansion of the universe, in contrast to (27) and (29). The difference from the dynamics obtained using transformations (29) is determined by the function $\sigma_k(t)$.

From the expression (82) for $k + 1$ element in chain of exact solutions

$$\dot{\sigma}_{k+1} = \exp\left[q_{k+1}\phi_{k+1}(t)\right], \tag{85}$$

using equation (80), one can obtain the following connection

$$\dot{\sigma}_{k+1} = \exp\left(\gamma\sqrt{n_k}q_{k+1}\ln\left[\frac{A_k\dot{\sigma}_k}{(C_k\sigma_k + B_k)^2}\right]\right). \tag{86}$$

As one can see, from (82)–(83), the transformations

$$\sigma(t) \to c_1\sigma(t) + c_2, \tag{87}$$

where $c_1 \neq 0$ and c_2 are some constants, don't change the character of the scalar field evolution and cosmological dynamic. Thus, one can define this function from connection (86) up to these transformations.

In general case, the connection between σ_{k+1} and σ_k form equation (86) can be expressed in quadratures only.

We also note, that under condition

$$\gamma\sqrt{n_k}q_{k+1} = 1, \quad \Rightarrow \quad q^2_{k+1} = \frac{1}{n_k}, \tag{88}$$

one can find from (86) the connections between σ_{k+1} and σ_k in explicit form

$$\sigma_{k+1} = -\frac{A_k}{C_k}(C_k\sigma_k + B_k)^{-1} + E_k, \tag{89}$$

where E_k is the constant of integration.

However, connections (89) lead to change of constants $\{A_k, B_k, C_k\} \Rightarrow \{A_{k+1}, B_{k+1}, C_{k+1}\}$ only in expression for the scalar field (82), and, therefore, such a transformations don't change the type of cosmological dynamics (83). Thus, to generate a new $(k+1)$–solutions in a chain it is necessary to solve equation (86) for each k-solutions.

6. Λ–chains of exact cosmological solutions

Now, we consider the examples of cosmological exact solutions which follow from the inflationary models with flat potential $V = \Lambda = const$ or cosmological constant Λ corresponding to de Sitter solutions [3]. Thus, the transformations (80)–(83) induce the deviations of the potential from a flat shape. Since, through inverse transformations towards (80)–(83) all these solutions are reduced to models with a cosmological constant, we will call them as Λ–*chains* of exact cosmological solutions.

6.1. Λ(exp)–chain of exact cosmological solutions

In such a chain for $i = 0$ from equations (24)–(25) one has following initial solutions

$$\phi_0 = 0, \qquad H_0 = \mu_0, \qquad a_0(t) = C_0\exp(\mu_0 t), \qquad V_0 = 3\mu_0^2, \tag{90}$$

which one can obtain from (19) and (25).

For the first order transformations $k = 0$ from (80)–(83) one has

$$\sigma_0(t) = t, \tag{91}$$

$$\phi_1(t) = \gamma\sqrt{n_0}\ln\left[\frac{A_0}{(C_0t + B_0)^2}\right], \tag{92}$$

$$H_1(t) = n_0\left(q_0^2\mu_0 + \frac{2C_0}{C_0t + B_0}\right) + \lambda_0, \tag{93}$$

$$a_1(t) = Q_1\left[C_0t + B_0\right]^{2n_0}\exp\left[(n_0q_0^2\mu_0 + \lambda_0)t\right]. \tag{94}$$

From equation (92) one has the dependence

$$(C_0t + B_0)^{-1} = \frac{\gamma}{\sqrt{A_0}}\exp\left(\frac{\phi_1}{2\gamma\sqrt{n_0}}\right). \tag{95}$$

After substituting dependence (95) into (93) one has

$$H_1(\phi_1) = \alpha_0 + \beta_0 \exp\left(\frac{\phi_1}{2\gamma\sqrt{n_0}}\right), \qquad \frac{dH_1(\phi_1)}{d\phi_1} = \omega_0 \exp\left(\frac{\phi_1}{2\gamma\sqrt{n_0}}\right), \tag{96}$$

where $\alpha_0 = n_0 q_0^2 \mu_0 + \lambda_0$, $\beta_0 = \frac{2\gamma n_0 C_0}{\sqrt{A_0}}$ and $\omega_0 = C_0\sqrt{\frac{n_0}{A_0}}$.

From equation (26) for $i = k+1$ one has

$$V_1(\phi_1) = (3\beta_0^2 - 2\omega_0^2)\exp\left(\frac{\phi_1}{\gamma\sqrt{n_0}}\right) + 6\alpha_0\beta_0 \exp\left(\frac{\phi_1}{2\gamma\sqrt{n_0}}\right) + 3\alpha_0^2. \tag{97}$$

Thus, as the result of the first-order transformations in $\Lambda(\exp)$–chain we obtain the exact solutions for exponential power-law (EPL) inflation [12].

In partial case $\alpha_0 = 0$ or $\lambda_0 = -n_0 q_0^2 \mu_0$ this model is reduced to the case of power-law inflation []. Also, for the case $3\beta_0^2 - 2\omega_0^2 = 0$ or $n_0 = 1/6$ one has the following potential

$$V_1(\phi_1) = 6\alpha_0\beta_0 \exp\left(\sqrt{\frac{3}{2}}\frac{\phi_1}{\gamma}\right) + 3\alpha_0^2. \tag{98}$$

For the second-order transformations with $k = 1$ from (80)–(83) we obtain

$$\sigma_1(t) = \alpha_1 \xi^{(2\beta_1 - 1)}, \tag{99}$$

$$\phi_2(t) = \gamma\sqrt{n_1}\,\ln\left[\frac{A_1 \xi^{2\beta_1}}{(C_1\alpha_1\xi^{(2\beta_1-1)} + B_1)^2}\right], \tag{100}$$

$$H_2(t) = n_1\left[q_1^2 H_1(t) + \frac{2C_1\xi^{2\beta_1}}{C_1\alpha_1\xi^{(2\beta_1-1)} + B_1}\right] + \lambda_1, \tag{101}$$

$$a_2(t) = Q_2\,[a_1(t)]^{n_1 q_1^2}\left[C_1\alpha_1\xi^{(2\beta_1-1)} + B_1\right]^{2n_1}\exp(\lambda_1 t), \tag{102}$$

where $\xi = \xi(t) = (C_0 t + B_0)^{-1}$, $\alpha_1 = \frac{\gamma (A_0)^{\beta_1}}{C_0(1-2\beta_1)}$ and $\beta_1 = \gamma\sqrt{n_0}q_1$.

As one can see, for $\beta_1 = 1$ or $B_1 = 0$, solutions (99)–(102) are reduced to (91)–(94) up to the choice of the constants. In general case, one can define the potential $V_2 = V_2(\phi_2)$ only in parametrical form.

For the case $\gamma q_1\sqrt{n_0} = 1/2$ from equation (86) one has

$$\sigma_1(t) = \omega_1 \ln u_1(t), \tag{103}$$

where $u_1(t) = C_0 t + B_0$ and ω_1 is an arbitrary constant.

For this function, from equations (82)–(83) one has

$$\phi_2(t) = \gamma\sqrt{n_1}\,\ln\left[\frac{A_1\omega_1 C_0}{u_1(t)(C_1\omega_1\ln u_1(t) + B_1)^2}\right], \tag{104}$$

$$H_2(t) = n_1\left[\frac{C_0}{2u_1(t)} + \frac{2C_1\omega_1 C_0}{u_1(t)(C_1\omega_1\ln u_1(t) + B_1)}\right] + \lambda, \tag{105}$$

$$a_2(t) = Q_2(C_0 t + B_0)^{\frac{n_1}{2}}\exp\left[\frac{\lambda}{C_0}(C_0 t + B_0)\right]\left[C_1\omega_1\ln(C_0 t + B_0) + B_1\right]^{2n_1}, \tag{106}$$

where all constant terms (shifts) in expression for the Hubble parameter are denoted as $\lambda \equiv \sum const$.

Thus, at the second order of transformations one has the modification of EPL dynamic by the third multiplier in expression (106). Such type of a model can be called as EPL–logarithmic inflation. Now, we find the corresponding potential for this inflationary model.

From equation (104) one has

$$u_1(t) = \frac{A_1 C_0}{4C_1^2\omega_1}\exp\left(-\frac{\phi_1}{\gamma\sqrt{n_0}}\right) W\left[\tilde{\alpha}_1 \exp\left(-\frac{\phi_1}{2\gamma\sqrt{n_0}}\right)\right]^{-2}, \tag{107}$$

where W denotes the Lambert function, $\tilde{\alpha}_1 = \tilde{\gamma}\frac{A_1 C_0}{\omega_1}\exp\left(\frac{B_1}{2C_1\omega_1}\right)$ and $\tilde{\gamma} = \pm 1$.

Thus, from (105) one has

$$H_2(\phi_2) = n_1 \varphi\, W(\varphi)\left[\tilde{\beta}_1 + \frac{\tilde{\omega}_1}{U(\varphi)}\right] + \lambda, \tag{108}$$

$$\frac{dH_2(\phi_2)}{d\phi_2} = \left(\frac{\sqrt{n_1}}{\gamma}\right)\frac{\varphi\, W^{3/2}(\varphi)}{1 + W^{1/2}(\varphi)}\left[\tilde{\beta}_1 + \frac{\tilde{\omega}_1}{U(\varphi)} - \frac{\tilde{\omega}_1 \omega_1 C_1}{U^2(\varphi)}\right], \tag{109}$$

where

$$\varphi \equiv \exp\left(\frac{\phi_2}{\gamma\sqrt{n_0}}\right), \quad W(\varphi) \equiv W\left[\tilde{\alpha}_1 \exp\left(-\frac{\phi_2}{2\gamma\sqrt{n_0}}\right)\right]^2, \tag{110}$$

$$U(\varphi) \equiv -C_1\omega_1 \ln\left[\left(\frac{4C_1^2\omega_1}{A_1 C_0}\right)\varphi W(\varphi)\right] + B_1, \quad \tilde{\omega}_1 = 8\frac{C_1^3\omega_1^2}{A_1}, \quad \tilde{\beta}_1 = 2\frac{C_1^2\omega_1}{A_1}. \tag{111}$$

Therefore, the potential for EPL–logarithmic inflation can be defined as follows

$$V_2(\phi_2(\varphi)) = 3\left(n_1\varphi\, W(\varphi)\left[\tilde{\beta}_1 + \frac{\tilde{\omega}_1}{U(\varphi)}\right] + \lambda\right)^2 - \frac{2n_1\varphi^2 W^2(\varphi)}{[1 + W^{1/2}(\varphi)]^2}\left[\tilde{\beta}_1 + \frac{\tilde{\omega}_1}{U(\varphi)} - \frac{\tilde{\omega}_1\omega_1 C_1}{U^2(\varphi)}\right]^2. \tag{112}$$

At the third order of transformations ($k = 2$) for the function (103) under condition $\gamma q_2\sqrt{n_1} = -1/2$ from equation (88) one has

$$\sigma_2(t) = \frac{2}{3}(C_0 t + B_0)^{\frac{3}{2}}\left[C_1\omega_1 \ln(C_0 t + B_0) + \frac{1}{3}(3B_1 - 2C_1\omega_1)\right]. \tag{113}$$

Also, at the third order of transformations for the function (103) under condition $\gamma q_2\sqrt{n_1} = -1$ from equation (88) one has the other type of the function $\sigma_2(t)$, namely

$$\sigma_2(t) = u_2(t)\ln^2(C_0 t + B_0) + v_2(t)\ln(C_0 t + B_0) + \alpha_2 t^2 + \beta_2 t + \omega_2, \tag{114}$$

where

$$u_2(t) = \frac{1}{2}\frac{\omega_1 C_1^2}{A_0}t^2 + \frac{\omega_1 C_1^2 B_0^2}{A_0 C_0^2}t + \frac{1}{2}\frac{\omega_1 C_1^2 B_0^2}{A_0 C_0^2}, \tag{115}$$

$$v_2(t) = \left(-\frac{1}{2}\frac{\omega_1 C_1^2}{A_0} + \frac{C_1 B_1}{A_0}\right)t^2 + \left(-\frac{\omega_1 C_1^2 B_0^2}{A_0 C_0^2} + 2\frac{C_1 B_1 B_0}{A_0 C_0}\right)t - \frac{1}{2}\frac{\omega_1 C_1^2 B_0^2}{A_0 C_0^2} + \frac{C_1 B_1 B_0^2}{A_0 C_0^2}, \tag{116}$$

$$\alpha_2 = \frac{1}{4}\frac{\omega_1 C_1^2}{A_0} - \frac{1}{2}\frac{C_1 B_1}{A_0} + \frac{1}{2}\frac{B_1^2}{A_0\omega_1}, \quad \beta_2 = \frac{1}{2}\frac{\omega_1 C_1^2 B_0}{A_0 C_0} - \frac{C_1 B_1 B_0}{A_0 C_0} + \frac{B_0 B_1^2}{\omega_1 A_0 C_0}. \tag{117}$$

The functions (113) and (114) give the different expressions for scalar field evolution $\phi_3 = \phi_3(t)$ and cosmological dynamics $H_3 = H_3(t)$ in explicit form from expressions (82)–(83) for $k = 2$, and the potential $V_3 = V_3(\phi_3)$ can be defined parametrically.

6.2. Λ(sinh)–chain of exact cosmological solutions

In such a chain for $i = 0$ from equations (24)–(25) one has following initial solutions

$$\phi_0(t) = \sqrt{\frac{2}{3}}\ln\left[\tanh\left(\frac{\alpha}{2}t\right)\right], \qquad H_0(t) = \frac{\alpha}{3}\coth(\alpha t), \qquad a_0(t) = C_0\sinh^{1/3}(\alpha t), \qquad V_0 = \frac{\alpha^2}{3}, \tag{118}$$

where α is a some constant.

From expression (80) for $k = 0$ one has

$$\sigma_0(t) = \int\left[\tanh\left(\frac{\alpha}{2}t\right)\right]^{\sqrt{\frac{2}{3}}q_0} dt. \tag{119}$$

The function $\sigma_0(t)$ can be obtained explicitly for different values of the constant q_0. Choosing $q_0 = \sqrt{\frac{3}{2}}m$, where m is an arbitrary constant, for the case $m = 1$ one has

$$\sigma_0(t) = \frac{2}{\alpha}\ln\left[\cosh\left(\frac{\alpha}{2}t\right)\right], \tag{120}$$

and for the other values $m \neq 1$ one can find the function $\sigma_0(t)$ from expression

$$\sigma_0(t) = \frac{2}{\alpha(1-m)}\left[\tanh\left(\frac{\alpha}{2}t\right)\right]^{m-1} + \int\left[\tanh\left(\frac{\alpha}{2}t\right)\right]^{m-2}dt. \tag{121}$$

Therefore, one has a set of exact solutions corresponding to different dynamics and different evolution of a scalar field at the first order of transformations (80)–(83).

Also, the expressions function (119) for each m induce the new chains of exact solutions of the second Einstein-Friedmann equation in higher orders of transformations (80)–(83).

6.3. Λ(cosh)–chain of exact cosmological solutions

In such a chain for $i = 0$ one has following initial solutions

$$\phi_0(t) = \sqrt{-\frac{2}{3}}\arcsin\left(\tanh\left(\alpha t\right)\right), \quad H_0(t) = \frac{\alpha}{3}\tanh(\alpha t), \quad a_0(t) = C_0\cosh^{1/3}(\alpha t), \quad V_0 = \frac{\alpha^2}{3}, \tag{122}$$

From expression (80) for $k = 0$ one has

$$\sigma_0(t) = \int\exp\left[q_0\sqrt{-\frac{2}{3}}\arcsin\left(\tanh\left(\alpha t\right)\right)\right]dt. \tag{123}$$

The function $\sigma_0(t)$ can be obtained explicitly for $q_0 = m\sqrt{6}$ and $q_0 = m\sqrt{\frac{3}{2}}$, where $m = \pm1, \pm2, \pm3, ...$

As the result, one has complex function $\sigma_0(t) = \sigma_0^{Re}(t) + \sigma_0^{Im}(t)$.

6.4. Λ(sin)–chain of exact cosmological solutions

In such a chain for $i = 0$ one has following initial solutions

$$\phi_0(t) = \sqrt{\frac{2}{3}}\mathrm{arctanh}\left(\cos(\alpha t)\right), \quad H_0(t) = \frac{\alpha}{3}\cot(\alpha t), \quad a_0(t) = C_0\sin^{1/3}(\alpha t), \quad V_0 = -\frac{\alpha^2}{3}, \tag{124}$$

From expression (80) for $k = 0$ one has

$$\sigma_0(t) = \int\exp\left[q_0\sqrt{\frac{2}{3}}\mathrm{arctanh}\left(\cos(\alpha t)\right)\right]dt. \tag{125}$$

The function $\sigma_0(t)$ can be obtained explicitly for $q_0 = \sqrt{\frac{3}{2}}m$ or $q_0 = m\sqrt{6}$, where $m = \pm1, \pm2, \pm3, ...$ Therefore, one has a set of exact solutions corresponding to different dynamics and different evolution of a scalar field at the first order of transformations (80)–(83).

Also, the expressions function (125) for each m induce the new chains of exact solutions of the second Einstein-Friedmann equation in higher orders of transformations (80)–(83).

6.5. Λ(cos)–chain of exact cosmological solutions

In such a chain for $i = 0$ one has following initial solutions

$$\phi_0(t) = \frac{1}{\sqrt{6}}\ln\left|\frac{1+\sin(\alpha t)}{1-\sin(\alpha t)}\right|, \quad H_0(t) = -\frac{\alpha}{3}\tan(\alpha t), \quad a_0(t) = C_0\cos^{1/3}(\alpha t), \quad V_0 = -\frac{\alpha^2}{3}, \tag{126}$$

From expression (80) for $k = 0$ one has

$$\sigma_0(t) = \int \left[\frac{1+\sin(\alpha t)}{\cos(\alpha t)}\right]^{\sqrt{\frac{2}{3}}q_0} dt. \tag{127}$$

The function $\sigma_0(t)$ can be obtained explicitly for $q_0 = m\sqrt{6}$ and $q_0 = m\sqrt{\frac{3}{2}}$, where $m = \pm 1, \pm 2, \pm 3, ...$ Therefore, one has a set of exact solutions corresponding to different dynamics and different evolution of a scalar field at the first order of transformations (80)–(83).

Also, the expressions function (127) for each m induce the new chains of exact solutions of the second Einstein-Friedmann equation in higher orders of transformations (80)–(83).

7. ϕ^2–chain of exact cosmological solutions

Now, we consider solutions for the extended Darbox class based on the solutions (75)–(79) for chaotic inflation with quadratic potential.

For the first order transformations $k = 0$ from (80) one has

$$\sigma_0(t) = \frac{A_0}{q_0\alpha}\exp\left[q_0(\alpha t + \beta)\right] = \frac{A_0}{q_0\alpha}e^{2u(t)}, \tag{128}$$

where $u(t) = \frac{1}{2}q_0(\alpha t + \beta)$.

From (81)–(83) one has

$$\phi_1(t) = -2\gamma\sqrt{n_0}\ln\left[\frac{C}{A_0}e^{u(t)} + \frac{B_0}{A_0}e^{-u(t)}\right], \tag{129}$$

$$H_1(t) = n_0\left[-\alpha q_0 u(t) + \frac{2C_0A_0}{B_0e^{-2u(t)} + C}\right] + \lambda, \tag{130}$$

$$a_1(t) = Q_1\exp\left(-\frac{t}{4}\left[\alpha n_0 q_0^2(\alpha t + 2\beta) - 4\lambda\right]\right)\left[Ce^{q_0(\alpha t+\beta)} + B_0\right]^{2n_0}, \tag{131}$$

where $C = \frac{A_0C_0}{q_0\alpha}$.

Without loss of generality we consider the function $\sigma_0(t)$ as follows

$$\sigma_0(t) = \exp\left[q_0(\alpha t + \beta)\right], \tag{132}$$

to generate the function $\sigma_1(t)$ at the second order of transformations ($k = 1$).

From equation (86) for $q_1 = \frac{m}{\gamma_0\sqrt{n_0}}$, where $m = 0, \pm 1, \pm 2, \pm 3, ...$ ($m \in Z$) for some specific value of m one has two types of solutions, namely

$$\sigma_1^{(+)}(t) = \sum_{s=-m}^{-2m+1}\omega_s v^s(t), \qquad \text{for } m \geq 1, \tag{133}$$

$$\sigma_1^{(-)}(t) = \varepsilon_m t + \sum_{s=-m}^{m}\mu_s v^s(t), \quad \text{for } m \leq 0, \tag{134}$$

where

$$v(t) \equiv C_0\exp\left[q_0(\alpha t + \beta)\right] + B_0, \tag{135}$$

and one has the exact expressions for the constants $\{\omega_s, \mu_s, \varepsilon_m\}$ for each s and m.

After substituting function (133)–(134) into (82)–(83) taking into account (130) one has the exact solutions for any value of the constant m.

Thus, we have an arbitrary number of exact cosmological solutions at the second order of transformations, which are defined by the choice of the value of constant m.

Conclusion

In this paper, we considered form-invariant transformations of the second equation of cosmological dynamics based on the Darboux transformations. Also, the Darboux transformations were combined with shifts and dilations. The result of this approach is the possibility of constructing an arbitrary number of exact solutions of the Einstein-Friedman equations.

As an example, we considered Λ-chains and ϕ^2-chain of exact cosmological solutions. The advantage of this approach is the ability to generate exact solutions corresponding to the complicated dynamics of the expansion of the early universe. Also, one can consider generalization of these exact solutions for the case multi-field cosmological models [14] and inflationary models based on the Einstein-Gauss-Bonnet gravity [15] as well.

Nevertheless, it is necessary to analyze cosmological solutions for the correspondence of the scalar field potential to the physical mechanisms of realization of the inflationary scenario, and the correspondence of the parameters of cosmological perturbations to the current observational constraints [16, 17].

Список литературы/References

1. D. Baumann, L. McAllister, *Inflation and String Theory*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 2015). https://doi.org/10.1017/CBO9781316105733.

2. S. D. Odintsov, V. K. Oikonomou, I. Giannakoudi, F. P. Fronimos, E. C. Lymperiadou, Recent Advances in Inflation, *Symmetry*, 2023, vol. 15, no. 9, p. 1701.

3. S. Chervon, I. Fomin, V. Yurov, A. Yurov, *Scalar Field Cosmology*, Series on the Foundations of Natural Science and Technology, Volume 13 (WSP, Singapur, 2019). https://doi.org/10.1142/11405.

4. S. V. Chervon, I. V. Fomin and A. Beesham, The method of generating functions in exact scalar field inflationary cosmology, *The European Physical Journal C*, 2018. vol. 78, no. 4, p. 301.

5. I. V. Fomin, S. V. Chervon, Exact and Approximate Solutions in the Friedmann Cosmology, *Russian Physics Journal*, 2017, vol. 60, no.3, pp. 427-440.

6. J. Martin, C. Ringeval, V. Vennin, Encyclopædia Inflationaris: Opiparous Edition, *Physics of the Dark Universe*, 2014, vol. 5-6, pp. 75-235.

7. V. M. Zhuravlev, S. V. Chervon, V. K. Shchigolev, New classes of exact solutions in inflationary cosmology, *Journal of Experimental and Theoretical Physics* , 1998, vol. 87, pp. 223-228.

8. A. V. Yurov, S. D. Vereshchagin, The Darboux transformation and exactly solvable cosmological models, *Theoretical and Mathematical Physics*, 2004, vol. 139, pp. 787-800.

9. A. V. Yurov, V. A. Yurov, Friedman versus Abel equations: A Connection unraveled, *Journal of Mathematical Physics*, 2010, vol. 51, 082503. https://doi.org/10.1063/1.3460856.

10. I. V. Fomin, S. V. Chervon, S. D. Maharaj, A new look at the Schrödinger equation in exact scalar field cosmology, *International Journal of Geometric Methods in Modern Physics*, 2018, vol. 16, no. 02, 1950022. https://doi.org/10.1142/S0219887819500221.

11. R. M. Hawkins and J. E. Lidsey, The Ermakov-Pinney equation in scalar field cosmologies, *Physical Review D*, 2002, vol. 66, 023523. https://doi.org/10.1103/PhysRevD.66.023523.

12. I. Fomin and S. Chervon, Exact and Slow-Roll Solutions for Exponential Power-Law Inflation Connected with Modified Gravity and Observational Constraints, *Universe*, 2020, vol. 6, p. 199.

13. A. D. Linde, Particle physics and inflationary cosmology, *Contemporary Concepts in Physics*, 1990, vol. 5, pp. 1-362.

14. S. V. Chervon, I. V. Fomin, E. O. Pozdeeva, M. Sami and S. Y. Vernov, Superpotential method for chiral cosmological models connected with modified gravity, *Physical Review D*, 2019, vol. 100, no. 6, 063522. https://doi.org/10.1103/PhysRevD.100.063522.

15. I. Fomin, Gauss–Bonnet term corrections in scalar field cosmology, *The European Physical Journal C*, 2020, vol. 80, no. 12, p. 1145.

16. N. Aghanim et. al., Planck 2018 results. VI. Cosmological parameters. *Astronomy* & *Astrophysics*, 2020, vol. 641, p. A6.

17. G. Galloni, N. Bartolo, S. Matarrese, M. Migliaccio, A. Ricciardone, N. Vittorio, Updated constraints on amplitude and tilt of the tensor primordial spectrum, *Journal of Cosmology and Astroparticle Physics*, 2023, vol. 04, 062. https://doi.org/10.1088/1475-7516/2023/04/062.

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Просьба ссылаться на эту статью следующим образом:

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Please cite this article in English as:

Fomin I. V. The extended class of exact cosmological solutions based on the Darboux transformations. *Space, Time and Fundamental Interactions*, 2025, no. 2, pp. 103–117.