УДК 539.12.01

НЕВОЗМОЖНОСТЬ СУЩЕСТВОВАНИЯ МАЙОРАНОВСКИХ СПИНОРОВ КАК ФИЗИЧЕСКИХ ЧАСТИЦ

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Майорановские спиноры играют важную роль в современных физических теориях. Большинство механизмов генерации массы нейтрино основаны на наличии майорановского массового члена в лагранжиане. В частности, механизм "качелей" генерации массы нейтрино. Майорановские решения уравнения Дирака существуют. Однако мы доказали, что массовый член лагранжиана майорановского спинора равен нулю. Мы доказали, что майорановские решения имеют нулевую энергию и импульс как в массивном, так и в безмассовом случае. Это означает, что майорановские спиноры не могут соответствовать физически существующим частицам.

Ключевые слова: масса нейтрино; майорановские спиноры; майорановские фермионы; майорановская масса.

THE IMPOSSIBILITY OF THE EXISTENCE OF MAJORANA SPINORS AS PHYSICAL PARTICLES

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Majorana spinors play an important role in modern physical theories. Most of the neutrino mass generation mechanisms are based on the presence of the Majorana mass term in the Lagrangian. In particular, the seesaw mechanism of neutrino mass generation. Majorana solutions of the Dirac equation exist. However, we have proven that Majorana spinor mass term of the Lagrangian is equal to zero. We have proven that Majorana solutions have zero energy and momentum for both the massive and massless cases. This means that Majorana spinors cannot correspond to physically existing particles.

Keywords: neutrino mass; Majorana spinors; Majorana fermions; Majorana mass.

PACS: 03.70.+k, 11.30.Er, 12.60.?i

DOI: 10.17238/issn2226-8812.2023.3-4.211-217

Introduction

Majorana spinors [1] play an important role in modern physical theories. Most of the neutrino mass generation mechanisms are based on the presence of the Majorana mass term in the Lagrangian. In particular, the seesaw mechanism of neutrino mass generation is a leading candidate for explaining the smallness of the neutrino mass [2]. This mechanism is based on the assumption of the existence of two types of neutrinos with a common mass matrix. When such a matrix is diagonalized, light and heavy neutrinos with Majorana masses appear.

Majorana solutions of the Dirac equation certainly exist. However, we have proven that for the Majorana spinor the mass term of the Lagrangian is equal to zero not only in the so-called c-theory [3], but also in the q-theory (second quantization theory) [4]. Therefore, there was a need to carefully study the properties of Majorana spinors in quantum field theory.

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1. CHARGE CONJUGATION OPERATION AND MAJORANA SPINORS

A Majorana spinor is a charge-self-conjugate (or charge-anti-self-conjugate) solution of the Dirac equation

$$\gamma^\mu i\partial_\mu \Psi = m\Psi \tag{1.1}$$

Such solutions were found by Majorana [1].

Dirac spinor Ψ_D is a superposition of charge-self-conjugate and charge-anti-self-conjugate Majorana spinors Ψ_{M1} and $i\Psi_{M2}$

$$\Psi_D = \frac{1}{\sqrt{2}}(\Psi_{M1} + i\Psi_{M2}) \tag{1.2}$$

In the Majorana representation of Dirac gamma matrices γ^μ , the charge conjugation operator $(\cdot)^c$ is the same as the complex conjugation operator $(\cdot)^*$, the gamma matrices are purely imaginary, and Majorana spinors Ψ_{M1} and Ψ_{M2} are real with respect to complex conjugation [1]

$$\begin{aligned} &(\cdot)^c = (\cdot)^* \,, \\ &\Psi^*_{M1} = \Psi_{M1} \,, \\ &\Psi^*_{M2} = \Psi_{M2} \,. \end{aligned} \tag{1.3}$$

In the Weyl (chiral) representation, the charge conjugation operation [5]

$$\begin{aligned} &(\cdot)^c = \eta_1\gamma^2(\cdot)^* \,, \\ &\Psi^c = (\cdot)^c\Psi = \eta_1\gamma^2\Psi^c \end{aligned} \tag{1.4}$$

in addition to complex conjugation, requires multiplication by $\eta_1\gamma^2$, where η_1 is an arbitrary phase factor. Usually it is considered equal to i [3], [6]. In what follows, we will assume that $\eta_1 = i$.

In general case

$$\begin{aligned} \Psi_{M1} &= \frac{1}{\sqrt{2}}(\Psi_D + \Psi^c_D) \,, \\ \Psi_{M2} &= \frac{1}{i\sqrt{2}}(\Psi_D - \Psi^c_D) \,. \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &\Psi^c_{M1} = \Psi_{M1} \,, \\ &\Psi^c_{M2} = \Psi_{M2} \,. \end{aligned} \tag{1.6}$$

2. LAGRANGIAN, HAMILTONIAN AND COMPONENTS OF MAJORANA SPINORS

As already said, we have proven that for the Majorana spinor the mass term of the Lagrangian is equal to zero not only in the so-called c-theory [3], but also in the q-theory (second quantization theory) [4]. Wherein

$$\overline{\Psi}_{M1}\Psi_{M1} = \overline{\Psi}_{M2}\Psi_{M2} = 0 \,. \tag{2.1}$$

The non-zero mass term $\mathscr{L}_M$ of the Lagrangian density arises only for the products of the fields of charge-self-conjugate and charge-anti-self-conjugate Majorana spinors, one of which is Dirac-conjugate

$$\mathscr{L}_M = -\frac{m}{2}\left(\overline{\Psi}_{M1} i\Psi_{M2} + (\overline{\Psi}_{M1} i\Psi_{M2})^+ \right). \tag{2.2}$$

The result is the Lagrangian density of the field of the Dirac spinor.

Left-chiral Dirac spinor in the Weyl representation can be represented as

$$\phi' = \begin{pmatrix} \phi'_1 \\ \phi'_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \phi' \\ 0 \end{pmatrix} \,, \tag{2.3}$$

where

$$\phi' = \begin{pmatrix} \phi'_1 \\ \phi'_2 \end{pmatrix} . \tag{2.4}$$

From (1.4) and (2.3) it follows [3], [5] that

$$\Psi_{LM} = \frac{1}{\sqrt{2}}(\Psi_L + \eta_1\gamma^2\Psi_L^*) = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi' \\ -\eta_1\sigma_2\phi'^* \end{pmatrix} . \tag{2.5}$$

A similar formula is obtained for the Majorana spinor obtained from the right-chiral Dirac spinor. For Ψ_{M2} , the expression is similar

$$\Psi_{M2} = \frac{1}{i\sqrt{2}}(\Psi_L - \eta_1\gamma^2\Psi_L^*) = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi'/i \\ -\eta_1\sigma_2(\phi'/i)^* \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi'' \\ -\eta_1\sigma_2\phi''^* \end{pmatrix} . \tag{2.6}$$

Thus, the most general expression for the components of the Majorana spinor is

$$\Psi_M = \frac{1}{\sqrt{2}}\begin{pmatrix} \phi \\ -\eta_1\sigma_2\phi^* \end{pmatrix} , \tag{2.7}$$

where

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} . \tag{2.8}$$

It is valid for Majorana spinors of both Ψ_{M1} and Ψ_{M2} types.

It follows from formula (2.7) that the two lower components (right-chiral) are expressed in terms of the two upper ones (left-chiral). Therefore, Majorana spinors have two times fewer independent components than Dirac spinors.

Matrix σ_2 rearranges the components with the spin projection up and down, and the ordering of the creation and annihilation operators for the left-chiral and right-chiral components of the Majorana spinor is the same. Because of this, Majorana spinors cannot have angular momentum projections and can only have helicity. In an implicit form, this was obtained in [5], and we indicate this explicitly. It follows from this that any physical system, which includes a Majorana spinor that does not interact with it, cannot also have a spin projection. Therefore, the Majorana spinor cannot exist as a physical particle.

The Lagrangian density $\mathscr{L}$ of the Majorana spinor Ψ_M is

$$\mathscr{L} = \frac{1}{2}\overline{\Psi}_M\gamma^\mu i\partial_\mu\Psi_M + \frac{1}{2}(\overline{\Psi}_M\gamma^\mu i\partial_\mu\Psi_M)^+ - m\overline{\Psi}_M\Psi_M . \tag{2.9}$$

Corresponding Hamiltonian density $\mathscr{H}$ is

$$\mathscr{H} = \frac{\partial\mathscr{L}}{\partial\dot{\Psi}_M}\dot{\Psi}_M - \mathscr{L} = \frac{\partial\mathscr{L}}{\partial\dot{\Psi}_M}\dot{\Psi}_M = \overline{\Psi}_M\gamma^0 i\partial_0\Psi_M = \Psi_M^+ i\partial_0\Psi_M . \tag{2.10}$$

In the Majorana representation, charge conjugation coincides with complex conjugation. That is why formulas

$$\begin{aligned} (\cdot)^c\Psi_M &= \Psi_M^* = \Psi_M , \\ (\cdot)^c\Psi_M^+ &= (\Psi_M^+)^* = \Psi_M^+ \end{aligned} \tag{2.11}$$

are satisfied in this representation.

From (2.10) and (2.11) it follows that

$$(\cdot)^c\mathscr{H} = -\mathscr{H}(\cdot)^c \tag{2.12}$$

in the Majorana representation.

Majorana spinor field operator Ψ_M is an eigenfunction of the charge conjugation operator $(\cdot)^c$. It follows from (2.12) that the Hamiltonian of the Majorana spinor cannot have nonzero eigenvalues. That is, the energy of the Majorana spinor must be identically equal to zero. In a similar way, one can prove that the spatial momentum of the Majorana spinor must be identically equal to zero. Therefore, the Majorana spinor cannot exist as a physical particle.

3. DIRAC AND MAJORANA SPINOR FIELD OPERATORS

Dirac spinor field operator is given by the standard formula [6]

$$\Psi_D = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E(p)}} \left(b_s(p) u_s(p) e^{-ip_\mu x^\mu} + d_s(p)^+ v_s(p) e^{ip_\mu x^\mu} \right), \tag{3.1}$$

where $b_s(p)$ is annihilation operator of the Dirac spinor with spatial momentum p and spin projection numbering s ($s = 1$ corresponds to the spin projection $+1/2$, $s = 2$ corresponds to the spin projection $-1/2$, or, as will be shown below, it is better to use helicity rather than spin projection), $d_s(p)^+$ is creation operator of the Dirac antispinor with spatial momentum p and spin projection (or helicity) corresponding to the index s, $u_s(p)$ and $v_s(p)$ are corresponding spinor columns.

We will use the Dirac representation, since the structure of $u_s(p)$ and $v_s(p)$ is simpler in it. We choose as a basis in the rest frame

$$u_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{3.2}$$

We have in the Dirac representation

$$i\gamma^2 = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

It is possible to define two more basis spinors $v_1(0)$ and $v_2(0)$ as

$$\begin{aligned} v_1(0) = u_1(0)^c = i\gamma^2 u_1(0)^* = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \\ v_2(0) = u_2(0)^c = i\gamma^2 u_2(0)^* = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}. \end{aligned} \tag{3.4}$$

In this case, relations

$$\begin{aligned} v_1(0)^c &= u_1(0), \\ v_2(0)^c &= u_2(0) \end{aligned} \tag{3.5}$$

are also satisfied.

Operator $(\cdot)^c$ commutes with Lorentz transformation generators $\gamma^{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$. Therefore, relations

$$\begin{aligned} u_s(p)^c &= v_s(p), \\ v_s(p)^c &= u_s(p) \end{aligned} \tag{3.6}$$

are satisfied for all p and $s = 1, 2$.

It follows from (3.4) that for negative-frequency states index s in $v_s(0)$ corresponds to the spin projection opposite to the spin projection for $u_s(0)$. That is, it corresponds to helicity, not spin projection.

Operators $b_s(p)$, $b_s(p)^+$, $d_s(p)$ and $d_s(p)^+$ have canonical anticommutation relations

$$\begin{aligned}
&\{b_i(p)^+, b_j(p')\} = \delta^i_j \delta(p-p')\,, \\
&\{d_i(p)^+, d_j(p')\} = \delta^i_j \delta(p-p')\,, \\
&\{b_i(p), b_j(p')\} = \{d_i(p), d_j(p')\} = \{b_i(p), d_j(p')\} = \{b_i(p)^+, d_j(p')\} = 0\,.
\end{aligned} \tag{3.7}$$

Moreover, $b_s(p)^+ + b_s(p)$, $b_s(p)^+ - b_s(p)$, $d_s(p)^+ + d_s(p)$ and $d_s(p)^+ - d_s(p)$ are generators of the infinite-dimensional Clifford algebra. It is known that generators of the Clifford algebra can always be chosen to be real with respect to the complex conjugation operation [7]. Therefore, we can set

$$\begin{aligned}
b_s(p)^* &= b_s(p)\,, \\
d_s(p)^* &= d_s(p)\,.
\end{aligned} \tag{3.8}$$

From (3.1), (1.5) and (3.8) it follows that

$$\begin{aligned}
\Psi_{M1} = \frac{1}{\sqrt{2}} \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E(p)}} (b_s(p) u_s(p) e^{-ip_\mu x^\mu} \;+\; d_s(p)^+ v_s(p) e^{ip_\mu x^\mu} \;+ \\
b_s(p) u_s(p)^c e^{ip_\mu x^\mu} \;+\; d_s(p)^+ v_s(p)^c e^{-ip_\mu x^\mu})\,.
\end{aligned} \tag{3.9}$$

Let us define operators

$$\begin{aligned}
a_s(p) &= \frac{b_s(p) + d_s(p)^+}{\sqrt{2}}\,, \\
a'_s(p) &= \frac{b_s(p) - d_s(p)^+}{i\sqrt{2}}\,.
\end{aligned} \tag{3.10}$$

Therefore, from (3.9), ((3.6) and (3.10) we obtain

$$\Psi_{M1} = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E(p)}} (a_s(p) u_s(p) e^{-ip_\mu x^\mu} \;+\; a_s(p) v_s(p) e^{ip_\mu x^\mu})\,. \tag{3.11}$$

Similarly, we obtain the formula for Ψ_{M2}

$$\Psi_{M2} = \sum_s \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E(p)}} (a'_s(p) u_s(p) e^{-ip_\mu x^\mu} \;-\; a'_s(p) v_s(p) e^{ip_\mu x^\mu})\,. \tag{3.12}$$

Operators (3.10) have canonical anticommutation relations

$$\begin{aligned}
&\{a_i(p)^+, a_j(p')\} = \delta^i_j \delta(p-p')\,, \\
&\{a'_i(p)^+, a'_j(p')\} = \delta^i_j \delta(p-p')\,, \\
&\{a_i(p), a_j(p')\} = \{a'_i(p), a'_j(p')\} = \{a_i(p), a'_j(p')\} = \{a_i(p)^+, a'_j(p')\} = 0\,.
\end{aligned} \tag{3.13}$$

It should be noted that in (3.11) and (3.12) the same operators $a_s(p)$ and $a'_s(p)$ appear as in the negative-frequency terms as in the positive-frequency terms. This is because the charge conjugation operator contains complex conjugation, but does not contain transposition. In this case, due to (3.8),

$$\begin{aligned}
a_s(p)^* &= a_s(p)\,, \\
a'_s(p)^* &= -a'_s(p)\,.
\end{aligned} \tag{3.14}$$

4. ENERGY AND MOMENTUM OPERATORS OF MAJORANA SPINORS

Similarly to how it was done in [6] for the Dirac spinors, we obtain energy P_0 and spatial momentum P_k operators of Majorana spinor Ψ_{M1}

$$\begin{aligned}
P_0 &= \sum_s \int d^3p\, p_0 (a_s(p)^+ a_s(p) - a_s(p)^+ a_s(p)) = 0\,, \\
P_k &= \sum_s \int d^3p\, p_k (a_s(p)^+ a_s(p) - a_s(p)^+ a_s(p)) = 0\,.
\end{aligned} \tag{4.1}$$

The same results are obtained for Ψ_{M2}

$$\begin{aligned} P_0 &= \sum_s \int d^3p\, p_0 (a'_s(p)^+ a'_s(p) - a'_s(p)^+ a'_s(p)) = 0\,, \\ P_k &= \sum_s \int d^3p\, p_k (a'_s(p)^+ a'_s(p) - a'_s(p)^+ a'_s(p)) = 0\,. \end{aligned} \tag{4.2}$$

Formulas (2.11), (2.12), (4.1) and (4.2) are true not only for massive but also for massless Majorana fields.

5. Discussion

Thus, we have proven that Majorana spinors have identically zero energy and momentum for both massive and massless cases. This means that Majorana spinors cannot correspond to physically existing particles.

The reason for the problems with spin projection, energy and momentum of Majorana spinors is due to the charge conjugation operation (1.3), (1.4). Majorana [1] and Kramers [8] defined operators (1.2) and (1.3) within the framework of the so-called c-theory, which preceded the theory of second quantization. For Dirac spinors, such a conjugation makes sense only in combination with the theory of “holes” in the Dirac Sea (fermions with negative energy). Majorana tried to construct a theory of fermions that did not require the concept of the Dirac Sea.

However, in quantum field theory, the theory of “holes” in the Dirac Sea was replaced by the use of fermion creation and annihilation operators. Therefore, in the theory of Dirac spinors, the positive-frequency components received the fermion annihilation operator as an additional factor, and the negative-frequency components received the antifermion creation operator.

Due to the fulfillment of equations (1.2) and (1.3), Majorana spinors have half the number of degrees of freedom than a Dirac spinor. Therefore, the approach that is suitable in the case of the Dirac fermion does not work in the case of the Majorana fermion.

It should be noted that charge conjugation operator C, defined by formulas (1.3) and (1.4), has another fundamental drawback. This operator contains complex conjugation and is therefore anti-unitary. But operator CPT must be anti-unitary, operator P must be unitary, operator T must be anti-unitary, and therefore operator C must be unitary [6], [9].

The Schwinger charge conjugation operator [10] is also used in the literature. In comparison with operators (1.3) and (1.4), it adds transposition of the creation and annihilation operators. However, it also contains complex conjugation and is therefore anti-unitary. Therefore, the question of constructing a consistent theory of charge-self-conjugate fermions and the see-saw mechanism based on it remains open.

Conclusion

Majorana spinors exist as solutions of the Dirac equation. However, we have proven that Majorana mass term in quantum field theory (QFT) must vanish and Majorana spinors can not have spin projections. Also, we have proven that the charge conjugation operator anticommutes both with energy and spatial momentum operators of Majorana spinors, that is why Majorana spinors cannot have non-zero energy and momentum. We confirmed this conclusion by deriving QFT formulas for the energy and momentum operators for both massive and massless Majorana spinors, by means of which we have proven that energy and momentum operators of Majorana spinors are identically equal to zero.

Thus, we have obtained several independent proofs that Majorana spinors cannot be physical particles. The results obtained do not imply the impossibility of constructing QFT theory of a truly neutral fermion or the impossibility of the seesaw mechanism. They mean that the question of constructing a consistent theory of charge-self-conjugate fermions and the see-saw mechanism based on it remains open.

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Просьба ссылаться на эту статью следующим образом:

Монахов В. В. Невозможность существования майорановских спиноров как физических частиц. *Пространство, время и фундаментальные взаимодействия.* 2023. № 3-4. С. 211–217.

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Please cite this article in English as:

Monakhov V. V. The Impossibility Of The Existence Of Majorana Spinors As Physical Particles. *Space, Time and Fundamental Interactions*, 2023, no. 3-4, pp. 211–217.