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ДВИЖЕНИЕ В ГРАВИТАЦИОННОМ ПОЛЕ ЧЁРНОЙ ДЫРЫ В СИНХРОННОЙ СИСТЕМЕ КООРДИНАТ

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Рассматривается движение пробного тела, или частицы, в гравитационном поле чёрной дыры, граничащей с тёмной материей. Статическое гравитационное поле предельно сжатой материи определяется путем решения уравнений Эйнштейна и Клейна-Гордона в синхронной системе координат. Предельно сжатое состояние материи в виде конденсата квантовой Бозе-жидкости энергетически более выгодно, чем вырожденный ферми-газ. Важным отличием от черных дыр Шварцшильда и Керра является отсутствие сингулярности в центре. В регулярном гравитационном поле, в зависимости от прицельного параметра, существуют траектории, ведущие сквозь "горизонт событий"внутрь чёрной дыры, а не только пролетающие мимо. При нулевой температуре в зависимости от парного взаимодействия бозонов, конденсат состоит из компонентов сверхтекучей и обычной (не сверхтекучей) квантовой жидкости. Задача о движении пробного тела внутри черной дыры решается аналитически в предельном случае, когда на фоне доминирующей гравитации, трением о не сверхтекучую компоненту Бозе-конденсата можно пренебречь.

Ключевые слова: Чёрная дыра, тёмная материя, синхронные координаты.

MOTION IN THE GRAVITATIONAL FIELD OF A BLACK HOLE IN A SYNCHRONOUS COORDINATE SYSTEM

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Motion of a test body, or a particle, in the gravitational field of a black hole bordering dark matter is considered. The static gravitational field of extremely compressed matter is determined by solving the Einstein and Klein-Gordon equations in the synchronous coordinate system. An extremely compressed state of matter in the form of a condensate of a quantum Bose liquid is energetically more favorable than a degenerate Fermi gas. An important difference from the Schwarzschild and Kerr black holes is the absence of a singularity in the center. In a regular gravitational field, depending on the impact parameter, there are trajectories leading through the “event horizon” into the black hole, and not just passing by. At zero temperature, depending on the pair interaction of bosons, the condensate consists of components of a superfluid and an ordinary (non-superfluid) quantum liquid. The problem of the motion of a test body inside a black hole is solved analytically in the limiting case when, against the background of dominant gravity, friction with the non-superfluid component of the Bose condensate can be neglected.

Keywords: Black hole, dark matter, synchronous coordinates.
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Introduction

In a synchronous coordinate system, a static regular solution to the system of Einstein and Klein-Gordon equations for the gravitational field of extremely compressed matter was found [1]. A maximally compressed black hole bordering dark matter claims to be the state that the gravitational collapse can lead to. [2].

In a synchronous reference frame ([3], §97) a spherically symmetric static metric

$$ds^2 = \left(dx^0\right)^2 - e^{2F_1\left(x^1\right)}\left(dx^1\right)^2 - e^{2F_2\left(x^1\right)}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \quad (0.1)$$

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contains two functions $F_1\left(x^1\right)$ and $F_2\left(x^1\right)$, depending on one coordinate x^1. Substitution

$$dr = e^{F_1\left(x^1\right)}dx^1, \quad r\left(x^1\right) = \int^{x^1} e^{F_1\left(x^1\right)}dx^1, \quad F_2\left(x^1\right) = F_2\left(r\left(x^1\right)\right) \tag{0.2}$$

changes (1) to the metric

$$ds^2 = c^2dt^2 - dr^2 + g_{22}\left(r\right)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right), \quad g_{22}\left(r\right) = -e^{2F_2(r)}. \tag{0.3}$$

containing only one metric function $F_2\left(r\right)$. It simplifies the solution to the system of Einstein and Klein-Gordon equations. In this case, the solution turns out to be more general, since it is valid for an arbitrary function $F_1(x^1)$ [1]. Unlike the Schwarzschild metric [4], the synchronous coordinate r is the real distance from the center.

In synchronous coordinates, as well as in Schwarzschild coordinates, in the equilibrium state of the extremely compressed Bose condensate, there are two gravitational radii r_g and r_h, on which the conditions of the "Existence and Uniqueness Theorem" ([5], §3) are not satisfied. There is a difference between these two coordinate systems. In the Schwarzschild coordinates ([3] formula (100.14)) the metric component $g^{11}\left(r\right) = 0$ at $r = r_g$ and $r = r_h$, so that in the interval $r_g < r < r_h$ the signature of the metric tensor is violated. And in the synchronous coordinate system (3) $g_{11}\left(r\right) = -1$ does not vanish anywhere. The metric signature does not change. In synchronous coordinates, the Einstein and Klein-Gordon equations are reduced to a second order system (and not to the fourth order as in the Schwarzschild ones). Taking into account the elasticity of the condensate in the $\lambda\psi^4$ model, the metric component $g_{22}\left(r\right)$ in (3) is derived analytically (formula (36) in [1]).

Considering the structure of the static equilibrium state of a supermassive black hole, it is natural to use the fact that gravity dominates over all other types of interactions. At the same time, today we have no reason to believe that a strong gravitational field affects the basic quantum properties of particles. That is, regardless of gravity, fermions remain fermions, and bosons remain bosons. It would seem that with dominant gravity, an ensemble of particles of a gravitating object can be considered a quantum ideal gas. However, without taking elasticity into account, the wave function of the Bose condensate diverges logarithmically at the center [2]. The singularity disappears due to the presence of arbitrarily weak repulsion of colliding particles [6].

1. Black hole and dark matter in synchronous coordinates

In a synchronous reference frame, a static solution to the Einstein and Klein-Gordon equations exists if matter is compressed by its own gravitational field to the ultrarelativistic limit $p = -\varepsilon/3$. The pressure p turns out to be negative because gravitational forces are aimed to compress the Bose condensate, and not to expand it. The equation defining the metric component $g_{22}\left(r\right) = -e^{2F_2(r)}$ is reduced to the form (formula (22) in [1]):

$$\frac{de^{F_2(r)}}{dr} = \sqrt{1 - \kappa\left|p\right|\left(e^{F_2(r)}\right)^2}. \tag{1.1}$$

Here $\kappa = \left(8\pi/c^4\right)k, \quad k = 6.67\times10^{-8}\,cm^3/\left(g\cdot\sec^2\right)$ − gravitational constant. The regular at the center solution to equation (4) is

$$g_{22}\left(r\right) = -e^{2F_2(r)} = -\frac{1}{\kappa\left|p\right|}\sin^2\left(\sqrt{\kappa\left|p\right|}r\right). \tag{1.2}$$

Solution (5) is unique only inside the sphere $0 < r < r_g$.

$$r_g = \frac{\pi}{2\sqrt{\kappa\left|p\right|}} \tag{1.3}$$

is the inner gravitational radius. At $r = r_g$ metric component

$$g_{22}\left(r_g\right) = -1/\kappa\left|p\right|. \tag{1.4}$$

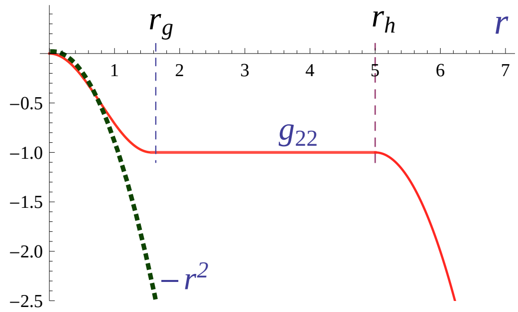


Рис. 1. Red line is metric component (1.7)

In the region $r > r_g$, the solution (5) to equation (4) with the boundary condition (7) is not unique [5]. The constant

$$g_{22}(r) = -\frac{1}{\kappa |p|} = -\left(\frac{2}{\pi} r_g\right)^2, \quad r \geq r_g \tag{1.5}$$

is also a solution to equation (4) with the boundary condition (7). In a spherical volume of radius r_h, the mass of the condensate $M = \frac{3c^2}{2k} r_h$ in the solution (1.5) is bigger than the mass in the solution (5) [1].

The sphere $r = r_h$ is the boundary of a black hole with dark matter. The gravitational properties of dark matter are adequately described using a longitudinal vector field [7]. The covariant divergence of the longitudinal vector field is a scalar that satisfies the same Klein-Gordon equation as the scalar wave function of the Bose condensate ([8] §30). However, the rest mass of a dark matter quantum can be many orders of magnitude less than the rest mass of Standard Model bosons. Using the condition of continuity of the function $F_2(r)$ and its derivative at the interface $r = r_h$, a solution to the system of Einstein and Klein-Gordon equations was found. It determines the component $g_{22}(r)$ of the metric tensor outside the black hole [1]:

$$g_{22}(r) = -\left(\frac{2}{\pi} r_g\right)^2 - (r - r_h)^2, \quad r > r_h. \tag{1.6}$$

In synchronous coordinates, in the $\lambda\psi^4$ model, the metric component $g_{22}(r)$ of the regular static gravitational field of interdependent black hole and dark matter is:

$$g_{22}(r) = \begin{cases} -\frac{4}{\pi^2} r_g^2 \sin^2\left(\frac{\pi}{2} \frac{r}{r_g}\right), & r < r_g, \\ -\frac{4}{\pi^2} r_g^2, & r_g \leq r \leq r_h, \\ -\frac{4}{\pi^2} r_g^2 - (r - r_h)^2, & r_h < r. \end{cases} \tag{1.7}$$

$g_{22}(r)$ is determined by two parameters – gravitational radii r_g and r_h. The graph of function $g_{22}(r)$ is the red line in Figure 1 [1]. The dotted line is $g_{22}(r) = -r^2$ in Schwarzschild coordinates.

The parameters $r_g = 1$ and $r_h = 5$ were chosen for clarity. In reality r_h can be many orders of magnitude greater than r_g.

Wave function of dark matter

$$\phi^r(r) = \frac{2}{\pi} \frac{\bar{\lambda} r_g}{\sqrt{\kappa}} \left[\left(\frac{2r_g}{\pi}\right)^2 + (r - r_h)^2\right]^{-1}, \quad r \geq r_h \tag{1.8}$$

decreases rapidly with distance from the black hole [1].

Below in this article I consider the motion of a test body, or a particle, in the gravitational field with metric (3), where the component $g_{22}(r)$ (1.7) is presented in Figure 1.

2. Test body in the gravitational field of a black hole and dark matter

2.1. General approach to a trajectory in synchronous coordinates

Consider the motion of a particle with mass m in a gravitational field in the synchronous reference frame (3). Let us choose the orientation of the metric so that the trajectory and the center are in the plane $\theta = \pi/2$. In a static spherically symmetric gravitational field $x^0 = ct$ and φ are cyclic coordinates. Accordingly, the associated energy E and angular moment M of a moving body are integrals of motion. The action $S(t, r, \varphi)$ satisfies the Hamilton-Jacobi equation (formula (9.19) in [3])

$$g^{ik}\left(\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k}\right) - m^2c^2 = 0$$

and allows separation of variables:

$$S(t, r, \varphi) = -\frac{E}{c}x^0 + M\varphi + S_r(r).$$

Metric (3) in the plane $\theta = \pi/2$

$$ds^2 = c^2dt^2 - dr^2 + g_{22}(r)\,d\varphi^2.$$

From the Hamilton-Jacobi equation

$$\frac{E^2}{c^2} - \left(\frac{\partial S_r}{\partial r}\right)^2 + \frac{M^2}{g_{22}(r)} - m^2c^2 = 0$$

we get the action

$$S(t, r, \varphi) = -Et + M\varphi \pm \int \sqrt{\frac{E^2}{c^2} - m^2c^2 + \frac{M^2}{g_{22}(r)}}dr. \tag{2.1}$$

The partial derivative of action (2.1) with respect to the angular moment M determines the trajectory of motion in polar coordinates r, φ:

$$\frac{\partial S}{\partial M} = \varphi \pm M \int \frac{dr}{g_{22}(r)\sqrt{\frac{E^2}{c^2} - m^2c^2 + \frac{M^2}{g_{22}(r)}}} = const. \tag{2.2}$$

By differentiating (2.1) with respect to energy E

$$\frac{\partial S}{\partial E} = -t \pm \frac{E}{c^2}\int \frac{dr}{\sqrt{\frac{E^2}{c^2} - m^2c^2 + \frac{M^2}{g_{22}(r)}}} = const \tag{2.3}$$

the dependence of the radius on time $r(t)$ is determined.

The dark matter wave function (1.8) rapidly decreases with distance from the center. Far from the black hole, the trajectory of a test particle is a straight line on the plane. At infinity, the distance ρ between the trajectory and the parallel straight line passing through the center is called the impact parameter. The impact parameter (impact factor) ρ is a constant connecting the conserved angular momentum M and energy E with the momentum P of the test body far from the black hole:

$$M = P\rho, \quad \frac{E^2}{c^2} - m^2c^2 = P^2, \quad \frac{E}{P} = \frac{c^2}{v}. \tag{2.4}$$

v is the speed of a test body along the trajectory. Taking into account (1.7) and (2.4), trajectory (2.2) is determined by three parameters of length dimension − gravitational radii r_g, r_h, and impact parameter ρ:

$$\varphi(r) \pm \rho \int \frac{dr}{\sqrt{g_{22}(r)\left(g_{22}(r) + \rho^2\right)}} = const. \tag{2.5}$$

The trajectory does not depend on the speed v of the test body. The distance to the center as a function of time (2.3) depends on the speed v:

$$t(r) \pm \frac{1}{v}\int \frac{\sqrt{g_{22}(r)}dr}{\sqrt{g_{22}(r) + \rho^2}} = const. \tag{2.6}$$

2.2. Trajectory outside a black hole

Outside a black hole, the test body moves through dark matter. To date, no direct interaction between ordinary matter and dark matter has been detected. We observe manifestations of dark matter only due to gravity. The test body outside the black hole moves in the common gravitational field of the black hole and dark matter.

According to (1.7), $g_{22}(r) = -\left(\left(\frac{2}{\pi}r_g\right)^2 + (r - r_h)^2\right)$ at $r > r_h$. The trajectory of the test body (2.5) outside the black hole

$$\varphi(r) = \pm\rho \int \frac{dr}{\sqrt{\left(\left(\frac{2}{\pi}r_g\right)^2 + (r - r_h)^2\right)\left(\left(\frac{2}{\pi}r_g\right)^2 + (r - r_h)^2 - \rho^2\right)}} + const, \quad r > r_h. \tag{2.7}$$

Outside the black hole $r > r_h$ the bracket $\left(\left(\frac{2}{\pi}r_g\right)^2 + (r - r_h)^2 - \rho^2\right)$ under the root vanishes at $r = r_{\min}$,

$$r_{\min} = r_h + \sqrt{\rho^2 - \left(\frac{2}{\pi}r_g\right)^2} = r_h + \rho\sqrt{1 - a^2}. \tag{2.8}$$

Here the dimensionless parameter

$$a = \frac{2r_g}{\pi\,\rho}. \tag{2.9}$$

The point on the trajectory closest to the center (turning point) (2.8), where the movement towards the center changes to the movement away from the center, exists provided that $a \le 1$. That is, the ratio of the impact parameter ρ to the internal gravitational radius r_g

$$\frac{\rho}{r_g} \ge \frac{2}{\pi} = 0.63662 \tag{2.10}$$

The trajectory of a test body does not touch the black hole under the condition $a < 1$.

The integral in (2.7) reduces to an elliptic integral of the first kind ([9], p. 918):

$$F(\xi, k) = \int_0^{\xi} \frac{d\alpha}{\sqrt{1 - k^2\sin^2\alpha}} = \int_0^{\sin\xi} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}}.$$

Namely, to the formula 3.152 6 on page 260 in [9]:

$$\int_b^u \frac{dx}{\sqrt{(x^2 + a^2)(x^2 - b^2)}} = \frac{F(\xi, s)}{\sqrt{a^2 + b^2}}, \quad \xi = \arcsin\sqrt{(a^2 + b^2)(a^2 + u^2)}, \quad s = \frac{a}{\sqrt{a^2 + b^2}}.$$

Outside the black hole, the trajectory is given by the formula:

$$\varphi(r) = \pm F\left(\arcsin\left[\rho\left(\left(\frac{2}{\pi}r_g\right)^2 + (r - r_h)^2\right)^{-1/2}\right], \frac{2r_g}{\pi\rho}\right) + const, \quad r > r_h. \tag{2.11}$$

However, it is easier to present a graph of the trajectory $r(\varphi)$ based on the equation

$$\frac{dr}{d\varphi} = \pm\frac{1}{\rho}\sqrt{\left[\left(\frac{2}{\pi}r_g\right)^2 + (r - r_h)^2\right]\left[\left(\frac{2}{\pi}r_g\right)^2 + (r - r_h)^2 - \rho^2\right]}, \quad r > r_h. \tag{2.12}$$

One can see that equation (2.12) is satisfied by an independent of φ constant $r(\varphi) = r_{\min}$ (2.8). It means that a circle with the radius $r_{\min}$ is a trajectory of the test body. I draw your attention to the fact that one would not immediately notice how the trajectory $r(\varphi) = r_{\min}$ is contained in formula (2.11). According to the existence and uniqueness theorem ([5], §3), this solution to equation (2.12) with a boundary condition $r(\varphi_0) = r_{\min}$ exists, but it is not unique even for an arbitrary φ_0. It is natural to look for other solutions to equation (2.12) with the same boundary condition $r(\varphi_0) = r_{\min}$ in the form

$$r(\varphi_0 + \delta\varphi) = r_{\min} + \gamma(\delta\varphi)^p \tag{2.13}$$

with a small $\delta\varphi$ different from zero. In the linear approximation, function (2.13) satisfies equation (2.12) if $p = 2$ and

$$\gamma\left(\gamma - \frac{1}{2}\sqrt{\rho^2 - \left(\frac{2}{\pi}r_g\right)^2}\right) = 0. \tag{2.14}$$

Solution with $\gamma = 0$ is the circle of radius $r_{\min}$: $r(\varphi) = r_{\min}$. A solution, different from this circle, is obtained by numerical integration of equation (2.12) with the boundary condition

$$r(\varphi_0 + \delta\varphi) = r_{\min} + \frac{1}{2}\sqrt{\rho^2 - \left(\frac{2}{\pi}r_g\right)^2}(\delta\varphi)^2, \quad \delta\varphi << 1. \tag{2.15}$$

It is convenient to choose the origin of the angular coordinate $\varphi = 0$ so that the turning point, where both solutions coincide, lies on the horizontal axis at the distance (2.8) from the center.

The expression under the radical (2.15) must be positive. If the ratio ρ/r_g is large, then the trajectory does not differ much from a straight line. In Figure 2a, the red circle is the boundary of the black hole with dark matter. Ratio $r_h/r_g = 10$. The blue circle is a trivial solution $r(\varphi) = r_{\min}$. The blue line tangent to the blue circle is the numerical solution to equation (2.12) with boundary condition (2.15). Ratio $\rho/r_g = 5$. With the decrease of ρ/r_g, both solutions approach the surface of the black hole. In Figure 2b $r_h/r_g = 10$, $\rho/r_g = 1$. As it approaches the lower boundary (2.10), the trajectory envelops the black hole. In Figure 2c $\rho/r_g = 0.651$. But this is not the limit. In the limit $\rho/r_g \to 2/\pi = 0.63662$ (2.10) the trajectory is completely wound around the black hole. At $\rho/r_g = 2/\pi$ both solutions merge into one circle on the surface of the black hole.

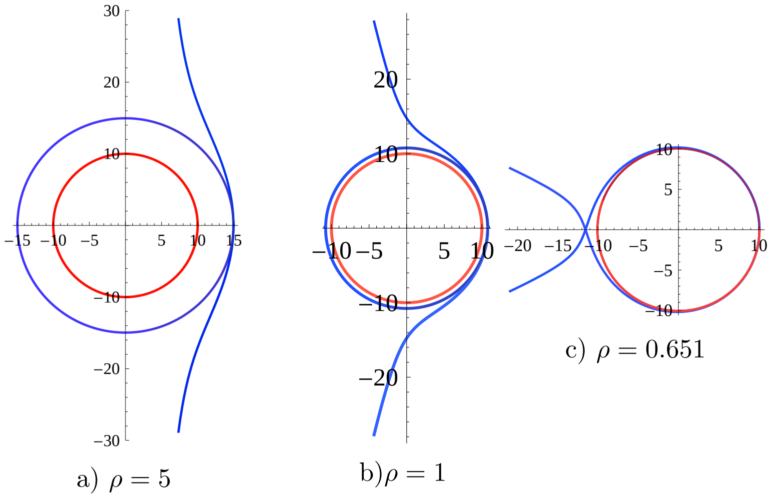


Fig. 2. Trajectories with parameters $\frac{\rho}{r_g} \geq \frac{2}{\pi}$, $r_g = 1$, $r_h = 10$

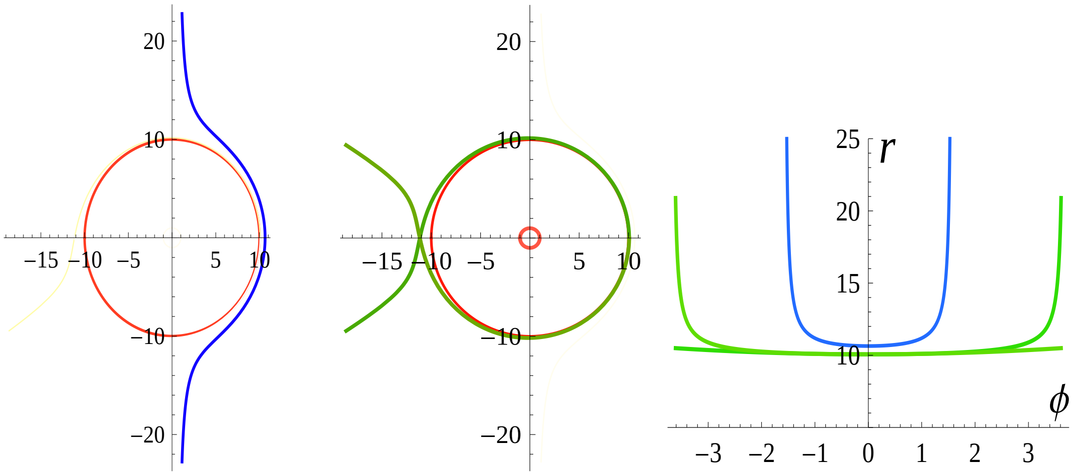
In reality, the internal gravitational radius r_g of a black hole can be many orders of magnitude smaller than the surface radius r_h. The presence of an internal gravitational radius r_g, no matter how small it may be, qualitatively changes the picture of motion of test bodies in the gravitational field of a black hole. If we put $r_g = 0$ in formulas (2.11) and (2.12), we get

$$\varphi(r) = \pm\arctan\sqrt{\frac{(r - r_h)^2}{\rho^2} - 1},$$

whence

$$r(\varphi) = r_h + \frac{\rho}{|\cos\varphi|}, \quad r > r_h. \tag{2.16}$$

For $r_g = 0$ and finite values $r_h > 0$, $\rho > 0$ we exclude the range of parameters $\rho/r_g \leq 2/\pi = 0.63662$ from consideration. The range of applicability for formula (2.16) is not only $r_g << r_h$, but also $r_g << \rho$.



a) $r_g = 0$ b) $r_g = 1$ c) $r_g = 0$ and $r_g = 1$

Fig. 3 The trajectories with $r_g = 0$ (blue lines) and $r_g = 1$ (green lines) . $r_h = 10$, $\rho = 0.64$

The trajectory with parameters $r_g = 0, r_h = 10, \rho = 0.64$ is presented in Figure 3a. For the same parameters $r_h = 10, \rho = 0.64$, but $r_g = 1$ the trajectory is shown in Figure 3b. For comparison, trajectories with $r_g = 0$ (blue curve) and $r_g = 1$ (green curve) are combined on a single graph in Figure 3c in Cartesian coordinates.

When $r_g = 0$ it follows from equation (2.12) that for any values $r_h > 0$ and $\rho > 0$ there is a turning point (2.8)

$$r_{\min} = r_h + \rho, \quad r_g = 0,$$

located outside a black hole. It means that at $r_g = 0$ there is no path along which anything could get inside a black hole. A "point-like" Schwarzschild's black hole (with a singularity at the center) [4] has only one gravitational radius r_h. For this reason, the surface radius r_h of a Schwarzschild black hole was considered to be an event horizon.

2.3. About the trajectory inside a black hole

Regularity at the center of a black hole without mass limitation occurs in the presence of an internal gravitational radius r_g [6]. When $r_g > 0$ the point closest to the center on the trajectory outside a black hole (2.8) exists if the parameter $a < 1$ (2.9). In this case, the impact parameter (impact factor) $\rho > \frac{2}{\pi} r_g$ (2.10). For $a > 1$, the impact factor is $\rho < \frac{2}{\pi} r_g$, and the bracket $\left(\left(\frac{2}{\pi} r_g\right)^2 + (r - r_h)^2 - \rho^2 \right)$ in the denominator (2.7) does not vanish outside the black hole. Therefore, there is no turning point outside the black hole, if $\rho < \frac{2}{\pi} r_g$. Naturally, trajectories with an impact parameter $\rho < \frac{2}{\pi} r_g$ inevitably lead inside the sphere $r = r_h$. In principle, this fact opens up the possibility of studying not only the surface, but also the internal physical properties of black holes.

The fate of a test particle inside a black hole is more complicated than outside. Outside a black hole, no interactions, other than the gravitational one, has been observed between ordinary and dark matter. Inside a black hole $r < r_h$, the test particle falls into the Bose condensate of ordinary (not dark) matter. Deriving the metric tensor component (1.7), I took into account that gravity dominates over all other interactions of black hole bosons. Bosons were considered as an extremely compressed ideal Bose gas. At zero temperature, the condensate of extremely compressed bosons can be in the state of a quantum liquid having the property of superfluidity ([8], Chapter 3). In an ideal superfluid liquid, the motion of a test body would occur without dissipation. But in an ideal liquid with no elasticity, the wave function of the condensate diverges logarithmically at the center [2]. Even arbitrarily small elasticity can be sufficient for regularity at the center. But in the presence of elasticity, a condensate is no longer an ideal liquid. Even at absolute zero, a Bose liquid contains both superfluid and normal components. Due to friction with the normal component, the motion of a test particle becomes dissipative. In this article, I consider the motion inside a black hole in the limiting case when the friction with the normal component can be neglected due to the dominating gravity. So, the test body moves with no violation of conservation of energy and angular momentum. I would like to note that the considered limit is the basis, deviations from which contain information about the properties of a black hole.

2.4. Motion within the spherical layer $r_g \le r \le r_h$

At $\rho < (2/\pi)\, r_g$ the metric component (1.7) in the spherical layer between the gravitational radii

$$g_{22}(r) = -\left(\frac{2}{\pi} r_g\right)^2, \quad r_g < r < r_h$$

does not depend on r. There is no turning point in the spherical layer $r_g \le r \le r_h$. $\varphi(r)$ (2.5) is a linear function:

$$\varphi(r) = \pm \frac{\pi}{2\sqrt{a^2-1}} \frac{r}{r_g} + \varphi_0, \quad r_g \le r \le r_h \tag{2.17}$$

The test particle inevitably falls inside the black hole if the parameter (2.9) $a = \frac{2r_g}{\pi\,\rho} > 1$. Moving from $r = r_g$ to $r = r_h$, the angular variable increases by

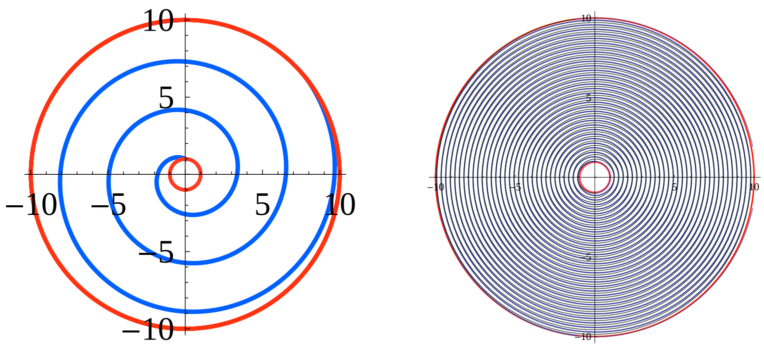
$$\varphi(r_h) - \varphi(r_g) = \frac{\pi}{2r_g} \frac{r_h - r_g}{\sqrt{a^2-1}}. \tag{2.18}$$

Trajectory $r(\varphi)$ (2.17)

$$r(\varphi) = \frac{2r_g}{\pi} \sqrt{a^2-1}\,(\varphi - \varphi_0), \quad r_g \le r \le r_h \tag{2.19}$$

is the spiral with a step

$$\Delta r = r(\varphi + 2\pi) - r(\varphi) = 4\sqrt{a^2-1}\,r_g. \tag{2.20}$$



a) $\rho = 0.5$ b) $\rho = 0.635$

Fig. 4. Spirals within the layer $r_g \le r \le r_h$. $r_g = 1$, $r_h = 10$

In Figure 4, there are two spiral trajectories within the layer $r_g < r < r_h$ between the red spheres. I chose gravitational radii $r_g = 1$ and $r_h = 10$ for clarity. In Figure 4a impact parameter $\rho = 0.5$. As ρ increases, the helix pitch (2.20) decreases. For comparison, at $\rho = 0.635$ (with $a = 1.00255$ very close to unity), spiral (2.19) with the same gravitational radii $r_g = 1$ and $r_h = 10$ is shown in Figure 4b.

2.5. Trajectory within the central area $r < r_g$

If we neglect dissipative processes during the motion of a test body through the Bose condensate of a black hole, then the dependence $\varphi(r)$ in the central area $r < r_g$ is determined by formulas (1.7) and (2.5):

$$\varphi(r) = \pm\frac{1}{a\rho}\int_{r_{\min}}^{r}\frac{dr}{\sin\left(\frac{\pi}{2}\frac{r}{\rho}\right)\sqrt{a^2\sin^2\left(\frac{\pi}{2}\frac{r}{r_g}\right)-1}}, \quad r<r_g. \tag{2.21}$$

Minimum distance $r_{\min}$ from the trajectory to the center

$$r_{\min} = a\rho\arcsin(1/a), \quad r<r_g. \tag{2.22}$$

$a=\frac{2}{\pi}\frac{r_g}{\rho}$ is the same parameter (2.9). For particles falling inside a black hole, $a>1$.

Integrating (2.21)

$$\varphi(r) = \pm\arctan\sqrt{\frac{a^2-1}{\cos^2\left(\frac{\pi}{2}\frac{r}{r_g}\right)}-a^2}, \quad 0<r<r_g,$$

we find the trajectory $r(\varphi)$ in the central region:

$$r(\varphi) = \frac{2}{\pi}r_g\arccos\sqrt{\frac{a^2-1}{a^2+\tan^2\varphi}}, \quad 0<r<r_g. \tag{2.23}$$

With increasing modulus of the angular coordinate $|\varphi|$ from zero to $\pi/2$, function $\tan^2\varphi$ in the denominator (2.23) varies from zero to infinity. The distance from the center $r(\varphi)$ increases from $r=r_{\min}$ (B.74) to $r=r_g$.

Trajectory (2.23) is not unique inside the sphere $r<r_g$. It becomes clear if formula (2.21) is presented in differential form:

$$\frac{dr}{d\varphi} = \pm a\rho\sin\left(\frac{\pi}{2}\frac{r}{r_g}\right)\sqrt{a^2\sin^2\left(\frac{\pi}{2}\frac{r}{r_g}\right)-1}, \quad r<r_g. \tag{2.24}$$

Obviously, the circle with radius $r_{\min}$ (B.74)

$$r(\varphi) = r_{\min} = const \tag{2.25}$$

is also a solution to equation (2.24). Equation (2.24) with the boundary condition $r(0)=r_{\min}$ is satisfied by two solutions (2.23) and (2.25). The blue circle in Figure 5 is the solution (2.25). The blue line connecting the top pole of the red circle to the bottom is the solution (2.23). Ratio $\rho/r_g=0.5$, $r_g=1$, radius of the blue circle (2.25) $r_{\min}=0.57508$.

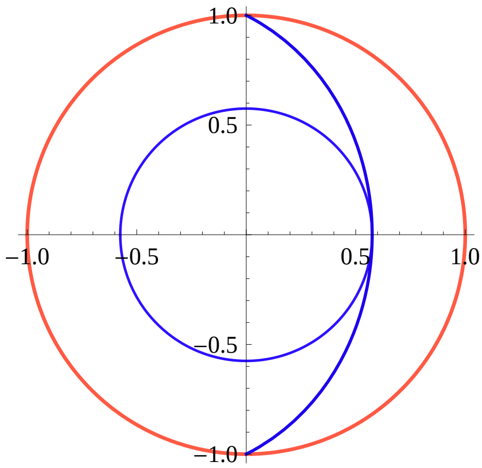


Fig. 5. Trajectories inside $r<r_g$. $r_g=1$, $\rho=0.5$.

In Figure 5, the blue line, connecting the upper and lower poles of the red circle, is the trajectory $r(\varphi)$ (2.23) inside a sphere of radius $r_g=1$. Impact factor $\rho=0.5$. The blue circle touching the blue line at point $r_{\min}$ (B.74) is also a possible trajectory inside the sphere $r<r_g$.

2.6. Time dependence $r(t)$ inside the sphere $r \le r_g$

I choose the beginning of time at the moment of passing the turning point (B.74) $t\left(r_{\min}\right) = 0$. Equation (2.6) with metric component (1.7)

$$g_{22}\left(r\right) = -\frac{4}{\pi^2} r_g^2 \sin^2\left(\frac{\pi}{2}\frac{r}{r_g}\right), \quad r < r_g$$

and with a boundary condition $t\left(r_{\min}\right) = 0$ has two solutions. First, time-independent rotation in a circle with a constant radius $r_{\min}$. And, secondly, the "schedule" at what time the test body is at the distance r from the center:

$$t\left(r\right) = \mp \frac{r_g}{v}\left[1 - \frac{2}{\pi}\arcsin\left(\sqrt{\frac{a}{a^2-1}}\right)\cos\left(\frac{2}{\pi}\frac{r}{r_g}\right)\right], \; r_{\min} < r < r_g. \tag{2.26}$$

Movement along the trajectory into the sphere $r < r_g$ begins from the gravitational radius r_g (moment $t\left(r_g\right) = -r_g/v$), penetrates deep into the turning point $r_{\min}$ (at the moment $t\left(r_{\min}\right) = 0$), and returns back to r_g (at the moment $t\left(r_g\right) = r_g/v$). The minus sign in (2.26) on the way towards the center, and the sign plus on the way back from the region $r < r_g$. The total time inside the sphere $r < r_g$ is $2r_g/v$, regardless of the impact parameter $\rho < (2/\pi)\, r_g$.

2.7. Complete trajectory

Trajectory (2.23) smoothly passes from the region $r < r_g$ to the region $r_g < r < r_h$,

$$\varphi\left(r\right) = \pm\frac{\pi}{2}\left(1 - \frac{r - r_g}{r_g\sqrt{a^2-1}}\right), \quad r_g < r < r_h, \tag{2.27}$$

if the constant $\varphi_0 = \pm\frac{\pi}{2}\left(1 - \frac{1}{\sqrt{a^2-1}}\right)$ in (2.17) . At the boundary of a black hole and dark matter, the spiral trajectory ends with coordinates $r = r_h, \; \varphi = \varphi\left(r_h\right)$ (C.1). Being a solution to equation (2.12) with the boundary condition $\varphi\left(r_h\right) = \pm\frac{\pi}{2}\left(1 + \frac{r_h - r_g}{r_g\sqrt{a^2-1}}\right)$, the trajectory smoothly continues outside the black hole.

An example of a complete trajectory of a particle in the dominant gravitational field of a black hole and dark matter is presented in Figure 6. $r_g = 1, \; r_h = 10, \; \rho = 0.5$. The paths to and from the center are marked in different colors (blue and green). Spheres with radii r_g and r_h are highlighted in red.

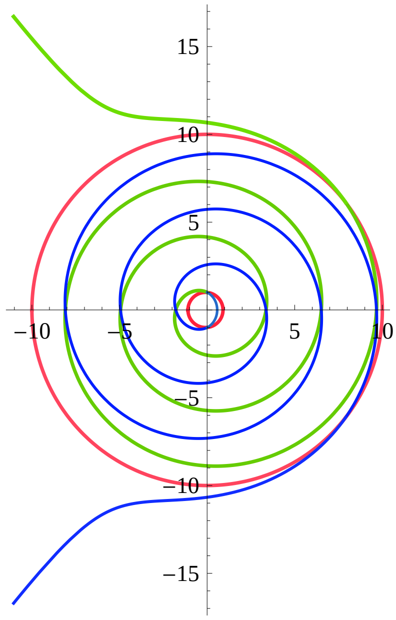


Fig. 6. Example of a complete trajectory. $r_g = 1,\ r_h = 10,\ \rho = 0.5$

The central sphere within the radius $r_g = 1$, magnified by the factor of 10, is shown in the previous Figure 5.

3. Real role of gravitational radii

In the Schwarzschild metric [4], a remote observer does not have the opportunity to reach the gravitational radius r_h of a black hole in a finite time. On this basis, for more than a hundred years, there is an opinion that a singularity at the center is inevitable. But nobody cared, because the singularity is located beyond the event horizon r_h. In fact, the distance to the gravitational radius r_h, infinite in time, is exclusively a property of the Schwarzschild coordinate system. In a synchronous coordinate system, the static gravitational field of a black hole and dark matter does not have a singularity in the center. Regularity in the center occurs due to existence of the internal gravitational radius $r_g < r_h$. The possibility or impossibility of a test body to get inside a black hole depends on the ratio of the impact parameter ρ and the internal gravitational radius r_g. At $\rho/r_g > 2/\pi$ the minimum distance of the trajectory to the center (2.8) exceeds the radius r_h of the black hole surface. A test body flies past a black hole. And vice versa: when $\rho/r_g < 2/\pi$ there is no turning point outside a black hole. In this case, a test body inevitably falls inside the black hole, no matter how small the finite radius $r_g > 0$ is.

In the Schwarzschild [4] and Kerr [10] metrics there is a singularity at the center, and there is no internal gravitational radius r_g. If $r_g = 0$ and the impact factor $\rho \neq 0$, than the turning point $r_{\min} > r_h$ is located outside the black hole. Moving of a test particle towards the center changes to moving away at $r = r_{\min}$, not reaching the surface of the black hole. In this sense, the gravitational radius r_h of the Schwarzschild and Kerr black holes appears to be an event horizon for a distant observer.

A regular at the center static solution to the system of Einstein and Klein-Gordon equations exists due to the arbitrarily low condensate elasticity [6]. The internal gravitational radius $r_g > 0$ in the regular solution depends on the elasticity of the condensate. In the Schwarzschild coordinate system, gravitational radii are separated by the fact that the component of the metric tensor $g^{rr} = 0$ at $r = r_g$ and $r = r_h$. The same solution in a synchronous coordinate system once again confirms that vanishing g^{rr} at $r = r_g$ and $r = r_h$ is the exclusive property of the Schwarzschild coordinate system. A distinctive property of invariance of gravitational radii r_g and r_h is the fact that in any frame of reference, solutions to the set of Einstein and Klein-Gordon equations with boundary conditions at $r = r_g$ and $r = r_h$ exist, but are not unique.

Conclusion

In the synchronous coordinate system, the existence of a regular static solution to the system of Einstein and Klein-Gordon equations is confirmed. This solution pretends to describe the extremely compressed state of a black hole surrounded by dark matter, to which gravitational collapse can lead. Moreover, with no limiting mass of a black hole. Unlike the singular in the center Schwarzschild's solution, regular solutions allow trajectories passing through the "event horizon" inside a black hole. Hence, a possibility opens up to study internal properties of black holes.

Список литературы/References

1. Meierovich B.E. Black Hole and Dark Matter in the Synchronous Coordinate System. *Journal of Experimental and Theoretical* Physics, 2023,Vol. No. 5, pp585-592.
2. Meierovich B.E. Static state of a black hole supported by dark matter. *Universe*, 2019, No. 5, 198.
3. Landau L.D., Lifshitz E.M. *The Classical Theory of Fields.* (*Nauka*, Moscow, 1973) (In Russian).
4. Schwarzschild K. Uberdas Gravitationsfeld eines Massenprunktes nach der Tinsteinschen Theorue. *Sitzungsberichte der Koniglich Ireuischen Academie der Wissenschaften: Berlin, Germany, 1916, pp.189-196.*
5. Pontryagin L.S. *Ordinary Differential Equations.* (Fizmatlit, Moscow, 1961).
6. Meierovich B.E. Guessing the Riddle of a Black Hole. *Universe* 2020, 6, 113.
7. Meierovich. B.E. Galaxy rotation curves driven by massive vector fields. Key to the theory of the dark sector. *Phys. Rev. D* 2013, 87, 103510.
8. Landau L.D., Lifshitz E.M. *Statistical Physics Part 2.* (Fizmatlit, Moscow, 2000).
9. Gradshtein I.S., Ryzik I.M. *Tables of integrals, summs, etc...* (Fizmatlit, Moscow, 1963).
10. Kerr R.P. Gravitational field of a spinning mass as an example of an algebraically special matrics. *Phys. Rev. Lett*, 1963. 11, 237-238.

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