УДК 537.8, 530.12

ОБОБЩЕННЫЕ МОДЕЛИ КАЛУЦЫ - КЛЕЙНА С ЛАГРАНЖИАНАМИ ГАУССА - БОННЕ

Кернер Р.[a,1]

[a] Университет Сорбонна, г. Париж, 75005, Франция.

Пятимерное обобщение эйнштейновской теории гравитации, впервые предложенное Т. Калуцей (1921) и улучшенное несколькими годами позже О. Клейном (1926), привело к модели Калуцы-Клейна, включающей электромагнетизм и гравитацию, и варианту теории гравитации Бранса-Дике, содержащему скалярное поле, взаимодействующее с метрическим тензорным полем. Однако ни одна из этих моделей не использовала возможности, открывающиеся при расширении вариационного принципа Эйнштейна-Гильберта за счет включения инварианта Гаусса-Бонне, который в 5 измерениях уже не является чистой дивергенцией и существенно модифицирует уравнения движения теории.

После напоминания основ модели Калуцы-Клейна, включая неабелев случай, мы даем краткий обзор многомерных космологических моделей со скалярными полями, порожденными калибровочными полями, вырожденными на структурной группе, включая обобщенный лагранжиан, содержащий член Гаусса-Бонне $R_{ABCD}R^{ABCD} - 4R_{AB}R^{AB} + R^2$.

Далее мы возвращаемся к 5-мерной модели Калуцы-Клейна, без скалярного поля и пренебрегая гравитацией, но с вариационным принципом, обогащенным членом Гаусса-Бонне. Это приводит в минковском пространстве-времени к интересному варианту нелинейной электродинамики. После обсуждения модифицированных уравнений Максвелла мы показываем, как может быть построен тороидальный солитон, и демонстрируем, что в нем проявляются наиболее существенные свойства электрона Дирака: электрический заряд, магнитный момент и спин. Он также предсказывает симметрию частица-античастица.

Ключевые слова: Теория Калуцы-Клейна, инварианты Гаусса-Бонне, нелинейная электродинамика, пучки волокон, космология в 10 измерениях.

GENERALIZED KALUZA-KLEIN MODELS WITH GAUSS-BONNET LAGRANGIANS

Kerner R.[a,1]

[a] Sorbonne University, Paris, 75005, France.

The five-dimensional generalization of Einstein's theory of gravitation proposed first by Th. Kaluza (1921) and improved a few years later by O. Klein (1926) has led to the Kaluza-Klein model incorporating electromagnetism and gravitation, and a variant of the Brans-Dicke theory of gravity, containing a scalar field interacting with metric tensor field. However, neither of these models did use the possibilities offered by the enlargement of the Einstein-Hilbert variational principle via including the Gauss-Bonnet invariant, which in 5 dimensions is no more a pure divergence, and modifies substantially the equations of motion of the theory.

After recalling the basics of the Kaluza-Klein model, including the non-abelian case. we give a short review of multi-dimensional cosmological models with scalar fields generated by gauge fields defined on the structural group, including the generalized lagrangian containing the Gauss-Bonnet term $R_{ABCD}R^{ABCD} - 4R_{AB}R^{AB} + R^2$.

Then we turn our attention back to the 5-dimensional Kaluza-Klein model, without scalar field and neglecting gravity, but with variational principle enriched by the Gauss-Bonnet term, This leads, in the Minkowskian space-time, to an interesting variant of non-linear Electrodynamics. After discussing the modified Maxwell's equations, we show how a toroidal soliton can be constructed, and show that it displays the most essential features of Dirac's electron: electric charge, magnetic moment, and spin. It also predicts the particle-anti particle symmetry.

Keywords: Kaluza-Klein theory, Gauss-Bonnet invariants, Non-linear Electrodynamics, Fibre Bundles, Cosmology in 10 dimensions.

[1]E-mail: richard.kerner@sorbonne-universite.fr

PACS: 03.50.-z, 03.50.Kk, 04.50.+h, 04.50.Kd
DOI: 10.17238/issn2226-8812.2023.3-4.166-187

1. Introduction

After the advent of the Relativity Theory proposed by Einstein i 1905, and its geometrical interpretation by Hermann Minkowski a few years later, the fully relativistic interpretation of electromagnetism was achieved. The Maxwell-Faraday theory was reformulated in terms of four-vectors and one and two-forms defined on the four dimensional space-time manifold.

The Kaluza-Klein theory was an attempt to unify classical field theories of gravitation and electromagnetism on the basis of the idea of the extension of the four-dimensional Minowskian spacetime by adding an extra spatial dimension, thus passing to a five-dimensional spacetime with pseudo-Euclidean metric $g_{AB} = \mathrm{diag}(+1, -1, -1, -1, -1)$. Curiously enough, the first to come with this idea was Gunnar Nordström (see [1]) who made his proposal of unification of gravitational and electromagnetic fields in 1914, one year before Einstein published his paper on General Relativity. In order to take gravitation into account Nordström added a fifth component to the electromagnetic vector potential. Note that he meant the Newtonian theory of gravity, represented by the scalar potential. Then the generalized Maxwell equations in five dimensions could be derived from a five-dimensional variational principle mimicking the lagrangian of the usual electromagnetism.

Introducing the five-dimensional vector potential

$$A_C = [A_\mu, A_5] = [A_\mu, \phi], \quad \text{with } \; B, C, .. = 1, 2, .., 5, \;\; \mu, \nu = 0, 1, 2, 3. \tag{1.1}$$

the Faraday-Maxwell field tensor could be generalized to five dimensons as follows:

$$F_{BC} = \partial_B A_C - \partial_C A_B, \quad \rightarrow \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \;\; F_{\mu 5} = -F_{5\mu} = \partial_\mu \phi - \partial_5 A_\mu. \tag{1.2}$$

$$\mathcal{L}_5 = -\frac{1}{4} F_{BC} F^{BC} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi. \tag{1.3}$$

The homogeneous set of Maxwell's equations is satisfied by virtue of the definition (1.2), giving two independent identities, the usual one, valid also in four dimensional version:

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0, \tag{1.4}$$

the extra set giving the following identities:

$$\partial_5 F_{\mu\nu} + \partial_\mu F_{\nu 5} + \partial_\nu F_{5\mu} = 0, \;\; \rightarrow \;\; \partial_5 F_{\mu\nu} = 0, \tag{1.5}$$

which reduces to a tautology due to the definition of Faraday's tensor; The last independent combination of three indices, $(5, \mu, 5)$, say, yields the following identity:

$$\partial_5 F_{\mu 5} + \partial_\mu F_{55} + \partial_5 F_{5\mu} = 0, \tag{1.6}$$

which is a tautology, too, because $F_{55} = 0$ and $F_{\mu 5} = -F_{5\mu}$ Therefore we cannot exclude in principle the dependence of the fifth component of our 5-dimensional vector potential (the scalar field ϕ included) on the fifth coordinate x^5. Let us however assume this for the sake of simplicity; then the differential system resulting from the variation of action with integrand given by (1.3) including the generalized current term $J^B A_B = j^\mu A_\mu + j^5 \phi$ is:

$$\partial_\mu F^{\mu\lambda} = -j^\lambda, \quad = -j^5, \tag{1.7}$$

reproducing the usual pair of Maxwell's equations with sources, which in appropriate units read:

$$\mathbf{rot}\; \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \;\text{ and }\; \mathrm{div}\mathbf{D} = \rho \tag{1.8}$$

the fifth component yielding the d'Alembert equation for the gravitational potential ϕ:

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \Delta\phi = \mu, \tag{1.9}$$

It is noticeable that in Nordström's 5-dimensional unification there is no interaction whatsoever between gravity and electromagnetism, they are described by totally independent potentials; on the other hand, the gravitational field is dynamical and can propagate with the speed of light. This was a step forward with respect to Newton's theory of gravitation, to which Nordström's model reduces in the case of a purely static field ϕ.

The 5-dimensional unification of electromagnetism and gravity was proposed by Th. Kaluza after Einstein published his General relativity paper, and was based on the generalized Einstein's equations, involving the metric tensor as dynamical variable. After an approval letter from Einstein to whom Kaluza sent his results in 1919, got an approval letter, he published them in 1921 ([2]).

Kaluza's paper contained an extension of general relativity to five dimensions, with a metric tensor of 15 components, out of which ten were identified with the four-dimensional spacetime metric, four components with the electromagnetic vector potential, and one component with a hypothetical scalar field (sometimes called the "dilaton". The 5-dimensional variational principle yields the 4-dimensional Einstein field equations, with the electromagnetic energy-momentum tensor as source on the right-hand side, the Maxwell equations for the electromagnetic field, and an extra equation for the scalar field. In order to simplify the theory, Kaluza introduced the "cylinder condition" hypothesis assuming that no component of the five-dimensional metric depends on the fifth dimension.

After the advent of Quantum Mechanics, Oskar Klein ([3]) gave Kaluza's classical five-dimensional theory a quantum interpretation. He assumed that the fifth dimension was compact and microscopic, to explain the cylinder condition. Klein suggested that the geometry of the extra fifth dimension could take the form of a circle, with the radius of 10^{-30} cm. More precisely, the radius of the circular dimension is 23 times the Planck length, which in turn is of the order of 10^{-33} cm.

Kaluza's and Klein's ideas seemed attractive enough to Einstein, who published his comment on five-dimensional theories in 1927 ([4]).

Classical theory was completed in the 40-ties and the full field equations including the scalar field were obtained almost simultaneously Yves Thiry ([5], [6]), Pasqual Jordan ([7]) and W. Scherrer ([8].

Jordan's work led to the scalar-tensor theory of Brans-Dicke ([9]), who were apparently unaware of Thiry's or Scherrer's papers.

The full expressions for the curvature tensors in the complete Kaluza equations were given by Coquereaux and Esposito-Farese ([16]).

What seems really amazing is that although the Gauss-Bonnet invariant of second order was known since a long time, the fact that it can be used as a non-trivial integrand for variational principle in mora than four dimensions, in particular in the Kaluza-Klein theory. The second-order Gauss-Bonnet invariant is the unique quadratic combination of Riemann tensor's components that under variation yields only second-order differential equations. It is a pure divergence in 4 dimensions, but starting from fivr dimensions its variation contributes to the second-order Einstein differential equations, although in a very non-linear way.

The general definition of Gauss-Bonnet invariant of order p is given in the followoing formula:

$$I_p = \frac{1}{2^p}\,\delta^{\mu_1\mu_2...\mu_p\nu_1\nu_2...\nu_p}_{\rho_1\rho_2...\rho_p\sigma_1\sigma_2...\sigma_p}\; R_{\mu_1\nu_1}{}^{\rho_1\sigma_1} R_{\mu_2\nu_2}{}^{\rho_2\sigma_2} ... R_{\mu_p\nu_p}{}^{\rho_p\sigma_p} \tag{1.10}$$

where the totally antisymmetric tensor of order n is defined as the anti-symmetrized prodict of n Kronecker's deltas:

$$\delta^{\mu_1\mu_2...\mu_n}_{\rho_1\rho_2...\rho_n} = \delta^{\mu_1}{}_{[\rho_1}\delta^{\mu_2}_{\rho_2}...\delta^{\mu_n}_{[\rho_n]} \tag{1.11}$$

An important feature of these invariants is that an invariant of order p, I_p, reduces to a pure divergence and do not contribute to the equations of motion under variation in dimensions lower than $2p$. Thus the first invariant, which is the Riemann scalar R, does not produce any equations in 2 dimensions: its integral over the entire manifold is constant and equal to $2\pi\chi$, where χ is the Euler-Poincaré characteristic (equal to 2 for a sphere, and 0 for a torus). The invariant I_2 is a pure divergence up to 4 dimensions, and I_3 is not a pure divergence only for manifolds whose dimension is higher than 6.

In 1971 Lovelock ([10]) presented an extension of Einstein's gravity to higher dimensions, with an enlarged lagrangian containing Gauss-Bonnet invariants. But to our knowledge, no one noticed that even in its original version the Kaluza-Klein five-dimensional theory can - and should - incorporate not only the usual Riemann curvature scalar, but also a non-trivial second order Gauss-Bonnet invariant. This possibility was never mentioned in modern expositions of the Kaluza-Klein theory that were written in early eighties by E. Witten ([11]), M. Duff ([13]), Th. Appelquist et al. ([12]) or by J. M. Overduin and P. S. Wesson ([14], [15]). The first extension of the Kaluza-Klein model incorporating the Gauss-Bonnet invariant of second order appeared in 1987 ([19])

2. Kaluza-Klein theory revisited

In the language of modern differential geometry, such a structure is called a *principal fibre bundle*, denoted by $P(M,G)$, where M denotes a differential manifold (in this case a pseudo-Riemannian space-time), and G is a compact and semi-simple Lie group (in this case the one-dimensional group $U(1)$, topologically equivalent to a circle. The canonical projection $\pi : P(M,G) \rightarrow M$ maps the points of $P(M,G)$ onto the points in M, $\pi(p) = x \in M$. The set of points in $P(M,G)$ that project on the same point $x \in M$ is called *a fibre*, and is isomorphic with the structure group G (here it is the $U(1)$ group: $\pi^{-1}(x) \sim U(1)$.

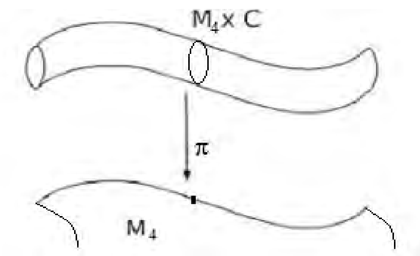

Рис. 1. Kaluza-Klein scheme.

The five-dimensional Kaluza-Klein space. The local coordinates are $x^A = (x^\mu, x^5)$; $A = 1,2,..5$, $\mu,\nu,.. = (0,i) = 0,1,2,3$, which under the projection π reduce to points in the Minkowski space-time: $\pi(x^A) = \pi(x^\mu, x^5) = (x^\mu) \in M_4$.

In its first version proposed by Th. Kaluza, the fifth dimension was just an extra space coordinate, the entire space being isomorphic with $M_4 \times R^1 \sim [ct, x, y, z, x^5] \sim M_5$, a five-dimensional Minkowski space. the five-dimensional metric can be regarded upon as the composition of two independent metrics, the 4-dimensional Minkowskian one and the "vertical" one defining an invariant scalar product in fibres, which in this case can be taken as $g_{55} = 1$.

If one assumes, as Oskar Klein proposed, that the fifth dimension is topologically closed, then it can be considered as a circle with a very small radius. The dependence on the fifth dimension of functions defined on the "compactified" space must be then periodic, admitting a Fourier-like decomposition:

$$f(x^\mu, x^5) = \sum_{k=0}^{\infty} a_k(x^\mu) e^{ikmx^5}. \tag{2.1}$$

with dim (m) $= cm^{-1}$. Then the eigenvalues of the fifth component of quantum momentum operator, $p_5 = -i\hbar\partial_5$ are integer multiples of mass m.

Let us recall the form of the Kaluza-Klein metric tensor in the absence of scalar field, $g_55 = -1$:

$$\tilde{g}_{AB} = \begin{pmatrix} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ A_\nu & -1 \end{pmatrix} \tag{2.2}$$

or more explicitly,

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + A_\mu A_\nu, \;\; \tilde{g}_{5\mu} = \tilde{g}_{\mu 5} = A_\mu, \;\; \tilde{g}_{55} = -1. \tag{2.3}$$

with A_μ functions of space-time variables, identified as the 4-vector potential.

The inverse metric tensor $\tilde{g}^{AB}$ has the following components in 5-dimensional space-time:

$$\hat{g}^{AB} = \begin{pmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & -1 + g_{\lambda\rho} A^\lambda A^\rho \end{pmatrix} \tag{2.4}$$

or more explicitly,

$$\hat{g}^{\mu\nu} = g^{\mu\nu}, \;\; \hat{g}^{5\mu} = \hat{g}^{\mu 5} = -A_\mu, \;\; \hat{g}^{55} = -1 + g_{\lambda\rho} A^\lambda A^\rho. \tag{2.5}$$

Nevertheless it turned out that this particular ansatz is still a solution to the full set of 15 equations, because in this case the last equation $R_{55} - g_{55} R = 0$ reduces to tautology $0 = 0$. This circumstance is often referred to as the “Kaluza-Klein miracle”

The explicit form of the remaining 14 equations in the 5-dimensional Einstein's general relativity theory is then:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{2} \eta^{\lambda\rho} F_{\mu\lambda} F_{\nu\rho} - \frac{1}{8}\, \eta_{\mu\nu} \eta^{\sigma\lambda} \eta^{\kappa\rho}\, F_{\sigma\kappa} F_{\lambda\rho} = -T_{\mu\nu} \tag{2.6}$$

$$R_{mu5} = \partial^\nu F^{\mu\nu} = 0, \quad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.7}$$

2.1. Adding the scalar field Φ

The full version of the Kaluza-Klein model englobes the gravitational field given 4-dimensional metric $g_{\mu\nu}(x)$, the electromagnetic field given by its 4-potential $A_\mu(x)$ and the scalar field $\Phi(x)$.

$$g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1), \;\; \mu, \nu, .. = 0, 1, 2, 3.$$

In this way we get the full set of 15 degrees of freedom present in the 5-dimensional Kaluza-Klein symmetric metric tensor $\hat{g}_{AB}, \;\; A, B, .. = 1, 2, ...5$.

In order to keep the fifth dimension spatial, g_{55} should be strictly negative; this is why we shall give it the form $g_{55} = -\Phi^2$.

Several particular situations can be chosen for study now. We can consider a case with scalar field only, without the electromagnetic one. This will lead to a variant of the tensor-scalar theory of gravitation, similar to the one proposed by Brans and Dicke. Another choice is the classical Kaluza-Klein model uniting gravitation and electromagnetism, but without scalar field. This amounts to suppressing one degree of freedom out of 15, leaving only 14 degrees of freedom, the 4-dimensional space-time metric $g_{\mu\nu}$ and the 4-vector potential encoded in the components $\hat{g}_{\mu 5} = \hat{g}_{5\mu}$ of the 5-dimensional metric.

Finally, we may consider the electromagnetic and scalar fields interacting in a flat Minkowskian space-time, the gravitation field considered as being negligible.

The five-dimensional metric with scalar field $\Phi(x)$ as the single degree of freedom remains diagonal:

$$g_{AB} = \text{diag}\left(+1, -1, -1, -1, -\Phi^2(x)\right). \tag{2.8}$$

In principle, the notation $\Phi(x)$ can mean the dependence of the scalar field not only on the space-time coordinates $(x^0 = ct, x^1, x^2, x^3)$ but also on the fifth coordinate x^5, so that in principle we may have not only $\partial_\mu \Phi \neq= 0$, but also $\partial_5 \Phi \neq= 0$.

However, supposing that the fifth dimension is the structural group $U(1)$ homeomorphic to a circle S^1, the dependence of Φ on x^5 can be only a periodic one:

$$\Phi(x^\mu, x^5) = \cos(n\, e\, x^5 + \delta) \cdot \phi(x^\mu), \;\; \text{so that} \;\; \partial_5^2 \Phi = -n^2 e^2 \Phi. \tag{2.9}$$

Let us derive the set of general formulas for metrics, connections and curvature in 5 dimensions, with all the 15 degrees of freesom present. The calculus in coordinates turns out to be quite complicated, but introduucing non-holonomic local frames simplifies the computations considerably.

The non-holonomic local frame is defined by means of the following set of 1-forms and vector fields: The 1-forms are:

$$\theta^\mu = dx^\mu, \quad \theta^5 = dx^5 + k\, A_\mu dx^\mu, \tag{2.10}$$

The dual vector fields, satisfying $\theta^A(\mathcal{D}_B) = \delta^A_B$ are:

$$\mathcal{D}_\mu = \partial_\mu - k\, A_\mu \partial_5, \quad \mathcal{D}_5 = \partial_5. \tag{2.11}$$

Let us introduce the following *transition matrices* U^A_B and $\overset{-1}{U}{}^B_C$ such that $\theta^A = U^A_B dx^B$, $\mathcal{D}_C = \overset{-1}{U}{}^D_C \partial_D$, so that we can write:

$$U^\mu_\nu = \delta^\mu_\nu, \quad U^\mu_5 = 0, \; U^5_\mu = kA_\mu, \; U^5_5 = 1;$$

$$\overset{-1}{U}{}^\mu_\nu = \delta^\mu_\nu, \; \overset{-1}{U}{}^5_\nu = -kA_\nu, \; \overset{-1}{U}{}^\mu_5 = 0 \; \overset{-1}{U}{}^5_5 = 1. \tag{2.12}$$

The metric tensor expressed in the non-holonomic frame can be deduced from the 5-dimensional length element squared, and becomes thus as follows

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu - \Phi^2 \left[dx^5 + k\, A_\mu dx^\mu\right]\left[dx^5 + k\, A_\nu dx^\nu\right] \tag{2.13}$$

leading to the following 5×5 matrix representation:

$$g^{AB} = \begin{pmatrix} g_{\mu\nu} + k^2\Phi^2 A_\mu A_\nu & -k\Phi^2 A_\nu \\ -k\Phi^2 A_\mu & -\Phi^2 \end{pmatrix} \tag{2.14}$$

The inverse matrix becomes then:

$$g^{BC} = \begin{pmatrix} g^{\nu\lambda} & kA_\lambda \\ kA_\nu & -\Phi^{-2} + k^2 A^\nu A^\lambda \end{pmatrix} \tag{2.15}$$

One easily checks that

$$g_{AB} g^{BC} = \delta^A_C.$$

The simplest and most elegant way to evaluate the connection coefficients and the components of the Riemann tensor is to use the non-holonomic frame θ^A and its dual basis of derivations (vector fields) $\mathcal{D}_B$, $A, B = 1, 2...5$.We need to know the commutators of non-holonomic derivations. We have:

$$[\mathcal{D}_A, \mathcal{D}_B] = C^E_{AB}\, \mathcal{D}_E, \tag{2.16}$$

where

$$C^5_{\mu\nu} = C_{\mu\nu 5} = -k\, F_{\mu\nu} = -k\, (\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{2.17}$$

We have then the connection coefficients in the non-holonomic basis:

$$\hat\Gamma^C_{AB} = \frac{1}{2}\hat g^{CE}\left[\mathcal{D}_A g_{BE} + \mathcal{D}_B g_{AE} - \mathcal{D}_E g_{AB}\right] + \hat g^{CE}\left[C_{EAB} + C_{EBA} - C_{BAE}\right] \tag{2.18}$$

where “hat” refers to the components with respect to the anholonomic frame.

The only non vanishing connection coefficients are then the following:

$$\hat\Gamma^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda}, \; \hat\Gamma^\mu_{\nu 5} = \hat\Gamma^\mu_{5\nu} = -\frac{1}{2}kF^\mu{}_\nu, \; \hat\Gamma^5_{\nu\lambda} = -\hat\Gamma^5_{\lambda\nu} = \frac{1}{2}kF_{\lambda\nu}, \tag{2.19}$$

The Riemann tensor expressed in a non-holonomic frame is:

$$\hat R^C_{AB\;\;D} = \mathcal{D}_A \hat\Gamma^C_{BD} - \mathcal{D}_B \hat\Gamma^C_{AD} + \hat\Gamma^C_{AF}\hat\Gamma^F_{BD} - \hat\Gamma^C_{BF}\hat\Gamma^F_{AD} - C^F_{AB}\hat\Gamma^C_{FD} \tag{2.20}$$

The Ricci tensor and the curvature scalar in 5 dimensions are calculated as usual,

$$\hat{R}_A D = \hat{R}^C_{AC \quad D}, \qquad \hat{R} = \hat{g}^{AB}\hat{R}_{AB}. \tag{2.21}$$

The resulting expression for the five-dimensional curvature is quite simple indeed:

$$\hat{R} = \overset{4}{R} - \frac{1}{4}\Phi^2 \, F_{\mu\nu}F^{\mu\nu} - \frac{2}{3\Phi^2}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi. \tag{2.22}$$

Considered as the integrand of a 5-dimensional variational principle, this Lagrangian density will lead to the following Einstein's equations when varying with respect to the metric only:

$$\hat{R}_{AB} - \frac{1}{2}\hat{g}_{AB}\hat{R} = 8\pi G\left[T^{(\Phi)}_{AB} + \frac{k^2}{16\pi G}T^{(F)}_{AB}\right] \tag{2.23}$$

where formally

$$T^{(\Phi)}_{AB} = \partial_A\Phi\partial_B\Phi - \frac{1}{2}\hat{g}_{AB}(\hat{g}^{CD}\partial_C\Phi\partial_D\Phi), \tag{2.24}$$

and

$$T^{(F)}_{AB} = F_{AC}F^C_{\ B} - \frac{1}{4}\hat{g}_{AB}(F_{CD}F^{CD}), \tag{2.25}$$

which in the case of the "n-th mode", i. e. the dependence Φ on x^5 in a periodic way, only to the space-time components different from zero:

$$T^{(\Phi)}_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}\hat{g}_{\mu\nu}\left[\hat{g}^{\lambda\rho}\partial_\lambda\Phi\partial_\rho\Phi - n^2e^2\Phi^2\right], \tag{2.26}$$

(where we neglected the mixed terms with $F_{\mu\nu}$) and where

$$T^{(F)}_{\mu\nu} = F_{\mu\lambda}F^\lambda_{\ \nu} - \frac{1}{4}\hat{g}_{\mu\nu}(F_{\lambda\rho}F^{\lambda\rho}), \tag{2.27}$$

Variation with respect to the scalar field Φ and the 4-vector potential A_μ lead to the following equations of motion:

$$\frac{1}{\Phi}\partial_\mu\left[\Phi \; F^{\mu\nu}\right] = 0, \tag{2.28}$$

and

$$(\Box\Phi + n^2e^2)\Phi = 0. \tag{2.29}$$

where the term n^2e^2 comes from the second derivative of Φ with respect to the circular coordinate x^5 and plays the role of a mass term for the Klein-Gordon scalar field equation.

3. Non-abelian generalization

An immediate and trivial generalization of the Kaluza-Klein model consists in adding more "external" dimensions, all of them repeating the same unit circle S^1 topology. In other words, instead of one cyclic dimension which can be also interpreted as a 1-dimensional Lie group $U(1)$, introduce a K-dimensional torus $T^K = S^1 \times S^1 \times ... \times S^1$.

The symmetry group of T^K is $[U(1)]^K$, the Cartesian product of K one-dimensional unitary groups $U(1)$. The corresponding Kaluza-Klein metric will be extended in a trivial way: let us label the extra dimensiona by $y^1, y^2, ...y^K$, and the set of all the coordinates by $x^B = (x^\mu, y^a), \quad A, B, .. = 1, 2, ..., 4 + K, \quad \mu, \nu = 0, 1, 2, 3, \quad a, b = 1, 2, ..., K$.

As a result, we shall get K distinct scalar fields $\Phi^a(x, y)$ and K distinct 4-potentials $A^b_\mu(x^\lambda)$, which will contribute separately to the action principle without any mutual interaction. In the absence of gauge fields and with the Minkowskian space-time the Kaluza-Klein multidimensional metric tensor would take on the Kasner metric form:

$$\hat{g}_{AB} = \text{diag}\,[+1,-1,-1,-1,-\Phi_1^2(x,y),-\Phi_2^2(x,y),...-\Phi_K^2(x,y)] \tag{3.1}$$

Let us suppose that the extra dimensions form a compact manifold of dimension N, endowed with a positive defined metric tensor g_{ab} When incorporated as a part of the global Kaluza-Klein metric, it will be taken with minus sign i order to comply with spatial nature of the extra dimensions. The extra dimensions can be thought of as a maximally symmetric manifold (an N-dimensional sphere) with its natural metric, or as a Lie symmetry group acting on it. The number of Killing vectors on the maximally symmetric space of dimension N is $K = N(N+1)/2$

Let us denote the K Killing vectors - left-invariant vector fields on the structural group - by $X_a, \;\; a = 1,2,...K$:

$$X_a = X_a \frac{\partial}{\partial y^b} \tag{3.2}$$

They satisfy the following commutation relations:

$$X_a^d \partial_d X_b^c - X_b^d \partial_d X_a^c = C_{ab}^f X_f^c \tag{3.3}$$

where the coefficients $C_{ab}^f = -C_{ba}^f$ are the structure constants of the Lie group generating the gauge symmetry. In what follows, we shall assume that the internal space is the group manifold itself.

The set of K 1-forms $\omega^b = \omega_c^b dy^c$ dual to the invariant vector fields is defined as follows:

$$\omega^b(X_a) = \omega_e^b X_a^e = \delta_a^b, \quad \text{also} \quad X_e^a \omega_b^e = \delta_b^a. \tag{3.4}$$

The ω^a are called the Maurer-Cartan forms. They satisfy the Maurer-Cartan equation:

$$\partial_a \omega_b^f - \partial_b \omega_a^f + C_{cd}^f \omega_a^c \omega_b^d = 0. \tag{3.5}$$

The invariant metric of the extra space is given by the Cartan-Killing symmetric tensor

$$g_{ab} = C_{ac}^d C_{bd}^c. \tag{3.6}$$

The overall metric tensor $g_{AB}, \;\; A,B.. = (\mu,b) = 1,2,..(K+4)$ in the non-abelian case is then:

$$\begin{pmatrix} g_{\mu\nu} + g_{ab} A_\mu^a A^b \nu & g_{ab}\omega_d^a A_\nu^b \\ g_{ab}\omega_c^a A_\mu^b & g_{ab}\omega_c^a \omega_d^b \end{pmatrix}, \tag{3.7}$$

$$\begin{pmatrix} g^{\nu\lambda} & -g^{\nu\rho} X_d^b A_\rho^d \\ -g^{\rho\lambda} X_c^a A_\rho^c & g^{ab} + g^{\mu\rho} X_c^a X_d^b A_\mu^c A_\rho^d \end{pmatrix}, \tag{3.8}$$

As in the 5-dimensional case, the calculus of the Riemann and Ricci tensors is made best in the non-holonomic frame.

The full set of expressions can be found in an article published in 1981 (R.K., *Ann. Inst. H. Poincaré*, **34**, p. 437-463) Here we give the resulting $4+K$ dimensional scalar curvature $\overset{(4+K)}{R}$ serving as integrand in the variational principle:

$$\overset{(4+K)}{R} = \overset{(4)}{R} - \frac{1}{4} g_{ab} g^{ab} - \frac{1}{4} g_{ab}^{\mu\lambda} g^{\nu\rho} F_{mu\nu}^a F_{\nu\rho}^b, \tag{3.9}$$

where $F_{\mu\nu}^a$ is the gauge field tensor given by:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + C_{bd}^a A_\mu^b A_\nu^d. \tag{3.10}$$

The resulting equations are similar as in the 5-dimensional case, with Einstein's equations given by

$$\overset{(4)}{R}_{\mu\nu} - \frac{1}{2} \overset{(4)}{g}_{\mu\nu} \overset{(4)}{R} = -\frac{8\pi G}{c^4} T_{\mu\nu} \tag{3.11}$$

with the energy-momentum tensor given by:

$$T_{\mu\nu} = g_{ab} g^{\lambda\sigma} F_{\mu\lambda}^a F_{\nu\sigma}^b - \frac{1}{4} g_{ab} g^{\mu\nu} g^{\lambda\rho} F_{\mu\lambda}^a F_{\nu\rho}^b. \tag{3.12}$$

and the gauge field equations are:

$$g^{\mu\nu}\; D_\mu F_{\nu\lambda}^a = g^{\mu\nu} \left[\partial_\mu F_{\nu\lambda}^a + C_{bd}^a A_\mu^c F_{\nu\lambda}^d \right] = 0 \tag{3.13}$$

4. Kaluza-Klein cosmology

A Generalized FRW metric can be easily introduced on the Kaluza-Klein manifold. In 1980 Chodos and Detwiler ([25]) proposed a Kasner-type cosmological solution in the 5-dimensional Kaluza-Klein space. The metric element for this model was

$$ds^2 = dt^2 - \sqrt{t}\left[dx^2 + dy^2 + dz^2\right] - \frac{1}{\sqrt{t}}\rho^2 d\chi^2. \tag{4.1}$$

where the last angular variable χ comes from the fifth cyclic dimension.

This metric can be generalized to more extra dimensions D; to be more precise:

$$ds^2 = dt^2 - \sum_{i=1}^{3} t^{2k_i}(dx^i)^2 - \sum_{a=4}^{3+D} t^{2k_a}(dy^a)^2 \tag{4.2}$$

satisfying the following conditions:

$$\sum_{i=1}^{3} k_i + \sum_{a=4}^{3+D} k_a = 1, \tag{4.3}$$

$$\sum_{i=1}^{3} k_i^2 + \sum_{b=4}^{3+D} k_b^2 = 1 \tag{4.4}$$

The Friedmann-Robertson-Walker metric can be naturally generalized if we assume that the extra space dimensions form a compact spherically symmetric manifold. Then the overall metric can be derived from the following line element squared:

$$ds^2 = dt^2 - R_d^2(t)\, g_{ij} dx^i dx^j - R_D^2(t) g_{ab} dy^a dy^b, \tag{4.5}$$

with two time-dependent scale factors, $R_d(t)$ for the space dimensions of our space-time, $d = 3$, and $R_D(t)$ for the D-dimensional internal D-dimensional compact space - most usually, a D-dimensional sphere.

This ansatz yields the following Ricci tensor:

$$R_{00} = -\left[3\frac{\ddot{R}_d}{R_d} + D\frac{\ddot{R}_D}{R_D}\right),$$

$$R_{ij} = \left[\frac{2k_d}{R_d^2} + \frac{d}{dt}\left(\frac{\dot{R}_d}{R_d}\right) + \frac{\dot{R}_d}{R_d}\left(3\frac{\dot{R}_d}{R_d} + D\frac{\dot{R}_D}{R_D}\right)\right] g_{ij},$$

$$R_{ab} = \left[\frac{2k_D}{R_D^2} + \frac{d}{dt}\left(\frac{\dot{R}_D}{R_D}\right) + \frac{\dot{R}_D}{R_D}\left(3\frac{\dot{R}_d}{R_d} + D\frac{\dot{R}_D}{R_D}\right)\right] g_{ab},$$

In 1985 D. Sahdev ([26]) obtained solutions of this system with several perfect fluids added on the right-hand side. The nice feature was that R_d was increasing with time, and R_D decreasing. However, instead of stabilizing at some small but finite value, as any reasonable physics would require, the internal radius R_D tended to zero.

Other models attempting to stabilize asymptotically the internal radius R_D were proposed by Matzner and Mezzacappa (see [23], by Copeland and Toms ([24]) and in six dimensions by Gleisser and Taylor (1985).

All those models were using the Einstein-Hilbert variational principle, with the integrand of the form

$$\delta \int \sqrt{\overset{(4+D)}{\hat{g}}{}_{AB}}\; \overset{(4+D)}{R}\; d^4x d^D y = 0$$

In 1988 we proposed a generalized non-abelian Kaluza-Klein model in 10 dimensions, with two gauge symmetry groups (B. Giorgini and R.Kerner, *Classical and Quantum Gravity*, **5** (1988), pp. 339-351) , which can be described as a double fibre bundle space, ([18])

$$P(P(V_4, SU(2)), SU(2)).$$

The lagrangian contained not only the usual Riemann scalar, but also the second-order and third-order Gauss-Bonnet invariants.

This situation gives place to three gauge fields, A^A_b, A^A_μ and A^a_ν, with indices $A, B = 1, 2, 3$ relating to the upper gauge geoup $SU(2)$, indices $a, b = 1, 2, 3$ relating to the lower gauge group $SU(2)$,and $\mu, \nu.. = 0, 1, 2, 3$ the space-time indices on V_4. The three field tensors become then:

$$F^B_{\mu\nu} = \partial_\mu A^B_\nu - \partial_\nu A^B_\mu + C^B_{CD} A^C_\mu A^D_\nu,$$

$$F^B_{cd} = \partial_c A^B_d - \partial_d A^B_c + C^B_{DE} A^D_c A^E_d,$$

$$F^b_{\mu\nu} = \partial_\mu A^b_\nu - \partial_\nu A^b_\mu + C^b_{cd} A^c_\mu A^d_\nu.$$

Looking for cosmological solutions, only the scalar multiplet $A^B_c(x, y)$ is of interest, the two vector potentials put to zero.

Decomposing A^B_c that is defined on the lower group space along the Maurer-Cartan forms:

$$A^B_c = \Phi^B_d(x^\mu)\omega^d_c, \tag{4.6}$$

the corresponding field tensor becomes:

$$F^E_{ab} = \left(C^E_{BD}\Phi^B_g\Phi^D_f - C^d_{gf}\Phi^E_d\right)\omega^g_a\omega^f_b. \tag{4.7}$$

due to the Maurer-Cartan identity fulfilled by ω^b_c.

The generalized FRW metric in 10 dimensions was as follows:

$$\overset{10}{g}_{\alpha\beta} = \text{diag}\left(1, -R^2(t)\delta_{ij}, -a^2(t)\delta_{ab}, -b^2(t)\delta_{AB}\right), \tag{4.8}$$

with $\alpha, \beta = 1, 2, ..., 10,\ i, j = 1, 2, 3,\ a, b = 1, 2, 3$ and $A, B = 1, 2, 3$.

The variational principle contained a linear combination of cosmological constant $\overset{10}{\Lambda}$, the scalar curvature $\overset{10}{R}$ and the 10-dimensional Gauss-Bonnet invariant $\overset{10}{GB}$.

The resulting differential equations determine the temporal behavior of three scale factors, the observable 3-dimensional space, and the two separate scale factors for two $SO(2)$ structural groups.

$$\int \sqrt{g}\ \left[\overset{10}{\Lambda} + \overset{10}{R} + \overset{10}{GB}\right] d^4x d^3\xi d^3\chi. \tag{4.9}$$

The equations are highly non-linear, but display several fixed points. Qualitative solutions were found with finite initial conditions for all three scale factors, leading asymptotically to Friedmann's solution for $R(t)$, while the internal scale factors behave differently: while $a(t)$ grows, $b(t)$ decreases, the exchange providing energy needed for the expansion of $R(t)$

5. Classical electrodynamics

Let us start by recalling the standard Maxwell's electromagnetism and fixing the notations. The simplest and the most elegant form of Maxwell's system is written in modern system of units as follows:

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \times \mathbf{E}, \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0, \tag{5.1}$$

$$\frac{\partial \mathbf{D}}{\partial t} + \mathbf{j} = \boldsymbol{\nabla} \times \mathbf{H}, \quad \boldsymbol{\nabla} \cdot \mathbf{D} = \rho, \tag{5.2}$$

It should be underlined that the pairs of fields ($\mathbf{E}$, $\mathbf{H}$) and ($\mathbf{D}$, $\mathbf{B}$) represent *different geometrical objects*. This can be better understood if we look at the integral form of Maxwell's equations:

$$-\frac{\partial}{\partial t} \oint_S \mathbf{B} \cdot \mathrm{d}\boldsymbol{\sigma} = \oint_{\partial S} \mathbf{E} \cdot \mathrm{dl}, \qquad \oint_{\partial V} \mathbf{B} \cdot \boldsymbol{\sigma} = 0. \tag{5.3}$$

$$\oint_S \left[\frac{\partial}{\partial t} \mathbf{D} + \mathbf{j} \right] \cdot \mathrm{d}\boldsymbol{\sigma} = \oint_{\partial S} \mathbf{H} \cdot \mathrm{dl}, \qquad \oint_{\partial V} \mathbf{D} \cdot \boldsymbol{\sigma} = Q. \tag{5.4}$$

Here S is a surface and ∂S its boundary, which is a closed line; V is a volume and ∂V is its boundary which is a closed surface. Correspondingly, we have vector fields and streams (2-forms). In the integral form of Maxwell's equations, the entities $\mathbf{E}$ and $\mathbf{H}$ are genuine vector fields which can be integrated along curves, whereas $\mathbf{B}$ and $\mathbf{D}$ are in fact 2-forms, defining *streams*.

The rate of change of fluxes of $\mathbf{D}$ and $\mathbf{B}$ through a surface is determined by the circulation of their conjugate fields $\mathbf{H}$ and $\mathbf{E}$ along the boundary, and vice versa.

A problem arises with number of equations versus number of functions: 8 equations for $4 \times 3 = 12$ components. The *constitutive relations* $\mathbf{E} = \mathbf{E}(\mathbf{D}, \mathbf{B})$ and $\mathbf{H} = \mathbf{H}(\mathbf{D}, \mathbf{B})$ reduce the number of variables to 6, thus making the system seemingly overdetermined.

Things become straightened up in a four-dimensional notation, with 4-vector potential defined as a vector in 4-dimensional space-time endowed wth Minkowskian metric $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$, $\mu, \nu, .. = 0, 1, 2, 3$

We assume that the 6 variables corresponding to the fields $\mathbf{E}$ and $\mathbf{B}$ are the 6 independent components of an antisymmetric 2-covariant tensor (a 2-form) $F_{\mu\nu} = -F_{\nu\mu}$, with $F_{0k} = E_k$, $F_{ik} = \epsilon_{ikm} B_m$, $i, k, m = 1, 2, 3$.

The Poincaré's Lemma states that if a 2-form - e.g. $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$ - is defined on an open subset of Minkowskian space-time M_4, then it is an exterior differential of some 1-form, then $A = A_\mu dx^\mu$:

$$F = dA \quad \to \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{5.5}$$

We have two independent relativistic invariant functions of Faraday's 2-form $F_{\mu\nu}$:

$$S = -\frac{1}{4} \eta^{\mu\lambda} \eta^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} = \frac{1}{2} \left(\mathbf{E}^2 - \mathbf{B}^2 \right), \tag{5.6}$$

$$P = -\frac{1}{8} \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} = F_{\mu\nu} \hat{F}^{\mu\nu} = \mathbf{E} \cdot \mathbf{B}, \tag{5.7}$$

with

$$\hat{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}$$

The choice of symbols is not accidental: S stands for "scalar", and P stands for "pseudo-scalar".

To ensure relativistic invariance, the variational principle should be derived from a Lagrangian depending on these two invariants, $\mathcal{L}(S, P)$. The equations of motion of the electromagnetic field form two groups: the homogeneous ones,

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0, \tag{5.8}$$

which are the consequence of the fact that $F = dA \ \to \ dF = d^2 A = 0$, and the equations resulting from variational principle applied to $\mathcal{L}$,

$$\partial_\mu G^{\mu\nu} = 0, \quad \text{with} \ \ G^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}} \tag{5.9}$$

$$G^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial S} F^{\mu\nu} + \frac{\partial \mathcal{L}}{\partial P} \hat{F}^{\mu\nu} \tag{5.10}$$

The dual Faraday tensor is given by definition:

$$G^{0i} = -G^{i0} = D^i, \quad G^{ik} = -G^{ki} = \epsilon^{ik}{}_l H^l \tag{5.11}$$

so the equations of motion become:

$$\frac{\partial \mathbf{D}}{\partial t} = \boldsymbol{\nabla} \times \mathbf{H}, \quad \boldsymbol{\nabla} \cdot \mathbf{D} = 0, \tag{5.12}$$

which coincide with Maxwell's second set of equations when the sources (the current density $\mathbf{j}$ and the charge density ρ) are put to zero.

The dynamical properties of the electromagnetic field are described by the energy-momentum tensor $T^{\mu\nu}$:

$$T^{\mu\nu} = F^{\mu}{}_{\lambda} G^{\lambda\nu} - \eta^{\mu\nu} \mathcal{L}, \tag{5.13}$$

$$T^{00} = \mathbf{E} \cdot \mathbf{D} - \mathcal{L}, \tag{5.14}$$

$$T^{0i} = (\mathbf{E} \times \mathbf{H})^i, \quad T^{i0} = (\mathbf{D} \times \mathbf{B})^i, \tag{5.15}$$

$$T^{ik} = -E^i D^k - H^i B^k + \delta^{ik}(\mathcal{L} + \mathbf{H} \cdot \mathbf{B}). \tag{5.16}$$

The energy-momentum tensor is symmetric and conserved:

$$T^{\mu\nu} = T^{\nu\mu}, \quad \partial_\mu T^{\mu\nu} = 0, \tag{5.17}$$

The proof uses the following identity:

$$F_{\mu\lambda} \hat{F}^{\lambda\nu} = \delta^{\nu}_{\mu} P, \tag{5.18}$$

The relations (5.17) result in the following conserved quantities

$$P^\mu = \int T^{\mu 0} \mathrm{d}\mathbf{r}^3, \quad M^{\mu\nu} = \int (x^\mu T^{\nu 0} - x^\nu T^{\mu 0}) \mathrm{d}\mathbf{r}^3 \tag{5.19}$$

Let us also note that the energy-momentum tensor could be obtained directly as

$$T_{\mu\nu} = \frac{\partial(\sqrt{\mid g \mid}\mathcal{L}}{\partial g^{\mu\nu}} \tag{5.20}$$

yielding the same expressions for the Hamiltonian T_{00} and the Poynting vector $P_k = T_{0k}$.

6. Kaluza-Klein Electrodynamics

Countless theories based on lagrangians depending on $F_{\mu\nu}F^{\mu\nu}$ and $(F_{\mu\nu}{}^* F^{\mu\nu})^2$ (the square is needed to keep the invariance under space reflections) can be produced if we lack a guiding principle to fix the form of the lagrangian. In the Kaluza-Klein theory, as well as in its improvements by P. Jordan and Y. Thiry was based on the Einstein-Hilbert variational principle in five-dimensional space, with lagrangian equal to R, the scalar curvature of the metric. This lagrangian is unique in four dimensions, because already the second invariant of the Riemann tensor,

$$I_2 = R_{ABCD} R^{ABCD} - 4\, R_{AB} R^{AB} + R^2 \tag{6.1}$$

turns out to be a pure divergence and does not modify the equations of motion.

The invariant (6.1) is the unique quadratic combination of the Riemann tensor leading under variation to the second-order equations. In five dimensions this invariant is no more a divergence, therefore there is no reason to exclude it in the full theory. This fixes the lagrangian in five dimensions, leaving the place for the arbitrariness only in the choice of one dimensional parameter.

This is the starting point for non-linear modification of the electrodynamics. In our calculations we shall discard the gravitational and scalar fields, both too weak to influence the behaviour of the electromagnetic field at short distances.

The invariant I_2 for the metric (2.2) is easily calculated and is found to be (discarding the pure divergence term equal to $\partial_\mu(F_{\rho\lambda}\partial^\mu F^{\rho\lambda}) - 2\partial^\nu(F_{\rho\lambda}\partial_\nu F^{\mu\lambda})$:

$$I_2 = \frac{3}{16}\ \left[(F_{\mu\nu}F^{\mu\nu})^2 - 2F_{\mu\lambda}F_{\nu\rho}F^{\mu\nu}F^{\lambda\rho}\right]. \tag{6.2}$$

For fixed Minkowskian metric $\eta_{\mu\nu}$ we can put $\sqrt{\mid g \mid} = 1$ and write the full lagragian as

$$\mathcal{L} = -\frac{1}{4}\ F_{\mu\nu}F^{\mu\nu} + \frac{3\varepsilon}{16e^2}\left[F_{\mu\nu}F^{\mu\nu})^2 - 2F_{\mu\lambda}F_{\nu\rho}F^{\mu\nu}F^{\lambda\rho}\right], \tag{6.3}$$

with ε a numerical parameter to be determined.

The equations of motion in vacuo are then

$$\partial_\lambda\left[F^{\lambda\rho} - \frac{3\varepsilon}{16e^2}(F_{\mu\nu}F^{\mu\nu})F^{\lambda\rho} + \frac{3\varepsilon}{e^2}F_{\mu\nu}F^{\lambda\mu}F^{\rho\nu}\right]. \tag{6.4}$$

The identities

$$\partial_\mu F_{\lambda\rho} + \partial_\lambda F_{\rho\mu} + \partial_\rho F_{\mu\lambda} = 0 \tag{6.5}$$

hold by definition (2.7), too:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Both lagrangian and equations of motion are more transparent when expressed by means of the fields $\mathbf{E}$ and $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$:

$$\mathcal{L} = \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \frac{3\varepsilon}{2e^2}(\mathbf{E}\cdot\mathbf{B})^2. \tag{6.6}$$

The new term contains only the square of the second invariant of the electromagnetic field. The full set of modified Maxwell's equations is:

$$\mathrm{div}\,\mathbf{B} = 0, \qquad \mathbf{rot}\,\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t},\ \ \mathrm{div}\,\mathbf{D} = -\frac{3\varepsilon}{e^2}\,\mathbf{B}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}),$$

$$\mathbf{rot}\,\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} + \frac{3\varepsilon}{e^2}\left[\mathbf{H}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} - \mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})\right]. \tag{6.7}$$

In what follows we shall use the units in which $c = 1$, and in which we can put in the vacuum $\mathbf{E} = \mathbf{D}$ and $\mathbf{H} = \mathbf{B}$. Therefore the equations in vacuum will be

$$\mathrm{div}\,\mathbf{B} = 0, \qquad \mathbf{rot}\,\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t},\ \ \mathrm{div}\,\mathbf{E} = -\frac{3\varepsilon}{e^2}\,\mathbf{B}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}),$$

$$\mathbf{rot}\,\mathbf{B} = \frac{\partial\mathbf{E}}{\partial t} + \frac{3\varepsilon}{e^2}\left[\mathbf{B}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} - \mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})\right]. \tag{6.8}$$

When ε is put equal to zero, the equations recover their usual Maxwellian form. Two other possibilities, up to a scale that can be incorporated in e^2, are $\varepsilon = +1$ or -1.

7. General properties

The non-homogeneous couple of equations,

$$\mathrm{div}\,\mathbf{E} = -\frac{3\varepsilon}{e^2}\,\mathbf{B}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}) \tag{7.1}$$

and

$$\mathbf{rot}\,\mathbf{B} = \frac{\partial\mathbf{E}}{\partial t} + \frac{3\varepsilon}{e^2}\left[\mathbf{B}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} - \mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})\right] \tag{7.2}$$

can be implemented by adding the charge density ρ to the right-hand side of (7.1) and the current density $\mathbf{j}$ to the right-hand side of (7.2)

However, even in the absence of these "external sources", the right-hand sides of the eqs. (7.1) and (7.2) behave like conserved induced charge and current densities; their conservation is independent of eventual other non-induced similar objects. As a matter of fact, let us compare:

$$\frac{\partial}{\partial t}(\mathrm{div}\,\mathbf{E}) = -\frac{3\varepsilon}{e^2}\frac{\partial \mathbf{B}}{\partial t}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}) - \frac{3\varepsilon}{e^2}\mathbf{B}\cdot\mathbf{grad}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} =$$

$$= -\frac{3\varepsilon}{e^2}(\mathbf{rot}\mathbf{E})\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}) - \frac{3\varepsilon}{e^2}\mathbf{B}\cdot\mathbf{grad}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} \tag{7.3}$$

and

$$\mathrm{div}\,\frac{\partial \mathbf{E}}{\partial t} = \mathrm{div}(\mathbf{rot}\mathbf{B}) - \frac{3\varepsilon}{e^2}\mathrm{div}\left(\mathbf{B}\,\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t}\right) - \frac{3\varepsilon}{e^2}\mathrm{div}(\mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})) =$$

$$= \frac{3\varepsilon}{e^2}(\mathrm{div}\mathbf{B})\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} - \frac{3\varepsilon}{e^2}\mathbf{B}\cdot\mathbf{grad}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} + \frac{3\varepsilon}{e^2}(\mathbf{rot}\mathbf{E})\cdot\mathbf{grad}(\mathbf{E}2\cdot\mathbf{B}). \tag{7.4}$$

because

$$\mathrm{div}\mathbf{B} = 0, \quad \mathbf{rot}(\mathbf{grad}f) = 0, \quad \mathrm{div}(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\mathbf{rot}\mathbf{a}) - \mathbf{a}\cdot(\mathbf{rot}\mathbf{b}),$$

therefore

$$\frac{\partial}{\partial t}\left[-\frac{3\varepsilon}{e^2}\mathbf{B}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})\right] + \mathrm{div}\left[\frac{3\varepsilon}{e^2}\mathbf{B}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} - \mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})\right] = 0. \tag{7.5}$$

We shall denote the induced charge density by ρ_{ind}:

$$\rho_{ind} = -\frac{3\varepsilon}{e^2}\mathbf{B}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}), \tag{7.6}$$

and the induced current density by $\mathbf{j}_{ind}$:

$$\mathbf{j}_{ind} = \frac{3\varepsilon}{e^2}\mathbf{B}\frac{\partial(\mathbf{E}\cdot\mathbf{B})}{\partial t} - \mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}) \tag{7.7}$$

with

$$\frac{\partial\rho_{ind}}{\partial t} + \mathrm{div}(\mathbf{j}_{ind}) = 0. \tag{7.8}$$

The theory does not need any non-induced charges if we can prove the existence of charged stable static solutions, (solitons localized in space). If we form the sum:

$$\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} \tag{7.9}$$

we shall easily find another conservation law:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2) + \frac{3\varepsilon}{e^2}(\mathbf{E}\cdot\mathbf{B})^2\right] = \mathrm{div}\,(\mathbf{E}\times\mathbf{B}) \tag{7.10}$$

The Poynting vector in this theory is the same as in the linear electrodynamics, whereas the energy density contains a new term, as compared with the classical theory:

$$\mathcal{H} = \frac{1}{2}(\mathbf{E}^2+\mathbf{B}^2) + \frac{3\varepsilon}{e^2}(\mathbf{E}\cdot\mathbf{B})^2 \tag{7.11}$$

Note that the parameter ε has to be positive, in order to ensure the positivity of the energy. From now on we shall set $\varepsilon = 1$, leaving only the coupling constant e^2 to be determined.

Whenever the fields $\mathbf{E}$ and $\mathbf{B}$ are orthogonal to each other, our system in vacuum (7.1, 7.2) coincides with Maxwell's equations. Such is the case of the electromagetic waves, which are also solutions to the equations (7.1, 7.2). Moreover, these solutions are stable with respect to perturbations. As a matter of fact, any deviation from the usual solution in which $\mathbf{E}$ is everywhere orthogonal to $\mathbf{B}$, leads automatically to the rise of the energy $\mathcal{H}$, ensuring stability.

8. Static solutions

Let us rewrite the equations (7.1 and 7.2 in the stationary case, when all the time derivatives vanish:

$$\text{div}\ \mathbf{B} = 0, \qquad \mathbf{rot}\ \mathbf{E} = 0,$$

$$\text{div}\ \mathbf{E} = -\frac{3}{e^2}\mathbf{B}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}), \quad \mathbf{rot}\ \mathbf{B} = -\frac{3}{e^2}\mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}). \tag{8.1}$$

It would be very interesting to obtain a static and non-singular solution of this system, having finite energy and behaving like a soliton.

This is excluded in the linear case, therefore, if such solution exists, both fields **E** and **B** must be different from zero and non-orthogonal at least in some finite domain of space. We should also impose the rapid enough vanishing of both fields at infinity. Spherical symmetry for **B** leads immediately to the singularity at the origin; so, if the condition div **B** is to be maintained everywhere, the lines of force of the field **B** have to be closed.

The lines of the local current

$$\mathbf{j}_{ind} = \mathbf{rot}\ \mathbf{B} = -\frac{3\varepsilon}{e^2}\ \mathbf{E}\times\mathbf{grad}(\mathbf{E}\cdot\mathbf{B})$$

must be closed, too. This suggests the axial symmetry in which the current would have only the azimuthal component, and the field **B** would be everywhere perpendicular to the azimuthal unit vector $\mathbf{e}_\varphi$ (in cylindrical coordinates $(\rho = \sqrt{x^2+y^2}),\ z,\ \varphi)$, i.e. **B** having its components along $\mathbf{e}_z$ and $\mathbf{e}_\rho$ only. Also the field **E** should have only the z and ρ components; then the Poynting vector $\mathbf{P} = \mathbf{E}\times\mathbf{B}$ will have only the azimuthal component.

Such a configuration has some remarkable symmetry properties:

The trilinear combinations on the right-hand sides of equations (8.1) produce induced charge and current densities.

The current having only the azimuthal component will produce magnetic field which at great distances is similar to that of a circular distribution of currents, i.e. the one of a magnetic dipole. At the same time, one can expect a non-vanishing charge concentration falling off quite rapidly with distance from the origin, at large distances **E** should be then similar to the Coulomb field of an electric point-like charge.

All these conditions put together lead to the following symmetry properties of the components **E** and **B**:

$$E_z(\rho, z) = -E_z(\rho, -z); \quad E_\rho(\rho, z) = E_\rho(\rho, -z), \tag{8.2}$$

and

$$B_z(\rho, z) = B_z(\rho, -z); \quad B_\rho(\rho, z) = -B_\rho(\rho, -z). \tag{8.3}$$

Let us evaluate the behaviour of charge and current distributions far away from the origin. We can take the field of a magnetic dipole and of concentrated charge as zeroth approximation satisfying Maxwell's equations, then insert them into the right-hand sides of eqs. (8.1) and compute the first corrections, supposing that the fields **E** and **B** develop as:

$$\mathbf{E} = \overset{(0)}{\mathbf{E}} + \frac{1}{e^2}\overset{(1)}{\mathbf{E}} + \ldots., \quad \mathbf{B} = \overset{(0)}{\mathbf{B}} + \frac{1}{e^2}\overset{(1)}{\mathbf{B}} + \ldots. \tag{8.4}$$

if we put

$$\overset{(0)}{\mathbf{E}} = \frac{Q\ \rho}{(\rho^2+z^2)^{\frac{3}{2}}}\mathbf{e}_\rho + \frac{Q\ z}{(\rho^2+z^2)^{\frac{3}{2}}}\mathbf{e}_z, \tag{8.5}$$

and

$$\overset{(0)}{\mathbf{B}} = \frac{3\mu\ \rho z}{4(\rho^2+z^2)^{\frac{5}{2}}}\mathbf{e}_\rho + \frac{\mu\ (2z^2-\rho^2)}{4(\rho^2+z^2)^{\frac{5}{2}}}\mathbf{e}_z, \tag{8.6}$$

where Q is the total charge, μ the total magnetic moment.

As the first correction, we obtain

$$\operatorname{div} \overset{(1)}{\mathbf{E}} = \frac{3\mu^2 Q}{8(\rho^2+z^2)^{\frac{11}{2}}}(\rho^2 + 10z^2), \tag{8.7}$$

and

$$\mathbf{rot}\ \overset{(1)}{\mathbf{B}} = \frac{3\mu Q^2\ \rho}{2(\rho^2+z^2)^{\frac{9}{2}}}\mathbf{e}_\varphi \tag{8.8}$$

which shows that the charge density falls off as R^{-9} and the current density as R^{-8} $(R = \sqrt{\rho^2+z^2})$, i.e. very fast indeed.

The lines of force of the field $\mathbf{B}$ form a family of closed curves which can be transformed into a family of circles by a suitable coordinate transformation; the toroidal coordinates are best adapted to describe the situation.

Let us introduce toroidal coordinates $(\mu,\ \eta,\ \phi)$:

$$\rho = \frac{a\ \sinh\mu}{\cosh\mu - \cos\eta}, \quad z = \frac{a\ \sin\eta}{\cosh\mu - \cos\eta}, \ \ \phi = \varphi, \tag{8.9}$$

with $0 \le \phi \le 2\pi$, $\quad 0 \le \eta \le 2\pi \quad$ and $0 \le \mu \le \infty$; $\ a$ is the constant of dimension of length fixing the scale; $\mu,\ \eta$ and ϕ are dimensionless.

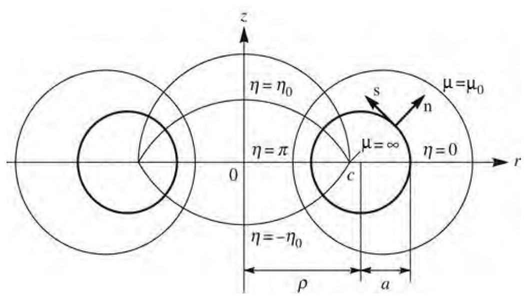

Рис. 2. Constant coordinate lines $\mu =$ Const. and $\eta =$ Const. in the (ρ, z)-plane.

A surface $\mu = \mu_0 =$ Const. is a torus with he external radius $a\ \coth\mu_0$ and internal radius $a/\sinh\mu_0$. When $\mu \to \infty$ it reduces to a circle of radius a in the (x, y)-plane. When $\mu \to 0$, the corresponding circle approaches the z-axis.

The lines of force of $\mathbf{B}$ coincide with circles $\mu =$ Const., i.e. in new coordinates (8.9)

$$\mathbf{B} = B_\eta(\mu,\eta)\ \mathbf{e}_\eta. \tag{8.10}$$

while $B_\mu(\mu,\eta) = 0$. This determines the dependence of $\mathbf{B}$ on η:

$$\text{as} \quad B_\mu(\mu,\eta) = (\mathbf{rot A})\cdot\mathbf{e}_\mu \quad \text{with} \quad \mathbf{A} = A_\phi(\mu,\eta)\mathbf{e}_\phi, \tag{8.11}$$

we have

$$B_\mu(\mu,\eta) = \frac{(\cosh\mu-\cos\eta)^2}{a\ \sinh\mu}\frac{\partial}{\partial\eta}\left(\frac{\sinh\mu}{\cosh\mu-\cos\eta}A_\phi\right) = 0. \tag{8.12}$$

Therefore

$$A_\phi(\mu,\eta) = (\cosh\mu - \cos\eta)\ G(\mu), \tag{8.13}$$

and

$$B_\eta(\mu,\eta) = -\frac{(\cosh\mu-\cos\eta)^2}{a\ \sinh\mu}\frac{\partial}{\partial\eta}(\sinh\mu\ G(\mu)) \tag{8.14}$$

with a yet unknown function $G(\mu)$.

Putting aside the problem of eventual singularity, we can at this point see quite well what the induced charge and current distributions look like. Consider one of the lines of force of $\mathbf{B}$, i.e. a circle $\mu =_m u_0$, $\phi = \phi_0$ in the (ρ, z) plane (Figure 1, left).

The symmetry properties of the field $\mathbf{E}$ impose the vanishing of its η-component for $z = 0$, i.e. for $\eta = 0$ or π, because $E_\eta(\eta) = -E_\eta(2\pi - \eta)$. On the other hand, $B_\eta(\eta) = B_\eta(2\pi - \eta) >$, so that the scalar product $\mathbf{E} \cdot \mathbf{B} = E_\eta B_\eta$ on the circle $\mu = \mu_0$ is an odd function of η (Figure 1, right).

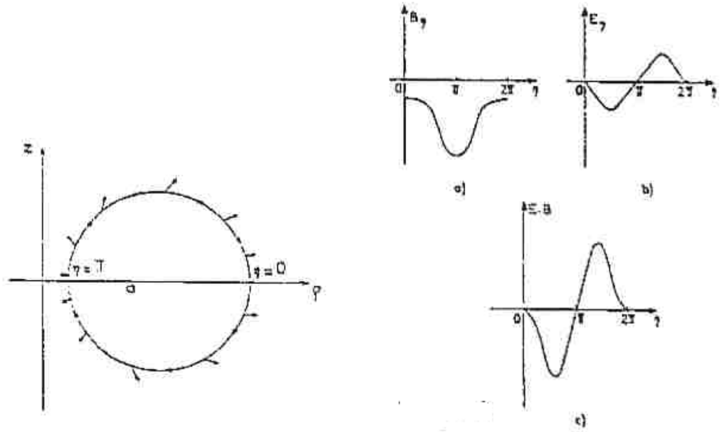

Рис. 3. Illustration for η

In order to obtain the charge distribution along this circle, we have to compute $-\mathbf{B} \cdot \mathbf{grad}(\mathbf{E} \cdot \mathbf{B})$, which reduces to the expression

$$-B_\eta \frac{(\cosh\mu - \cos\eta)}{a} \frac{\partial}{\partial\eta} (\mathbf{E} \cdot \mathbf{B}) \,. \tag{8.15}$$

The corresponding functions are displayed in Figure 3:

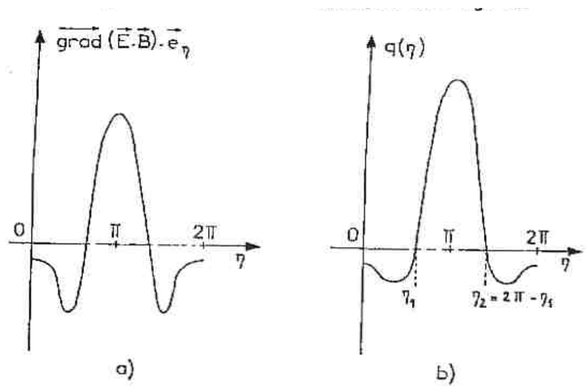

Рис. 4. a) The projection of $\mathbf{grad}(\mathbf{E} \cdot \mathbf{B})$ on the unit vector $\mathbf{e}_\eta$ as a function of η; b) The charge density distribution $q(\eta)$ as function of η

The charge density changes its sign between η_1 and $\eta_2 = 2\pi - \eta_1$. This phenomenon describes vacuum polarization: if at the core of the static solution there is an accumulation of charge density of a given sign, it must be surrounded by a cloud of charge density of opposite sign.

The value of η_1 at which the change of sign occurs depends on the line (i.e. the value of μ). Reproducing similar reasoning for all circles $\mu =$ Const. we obtain the picture of the overall charge density (Figure 4):

The strongest vacuum polarization is on the z-axis and in the symmetry plane (x, y), around the axially symmetric charge distribution at the core. If at any point of this distribution we wanted to

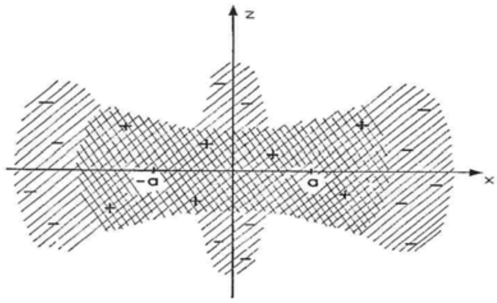

Рис. 5. The cross-section $(x, z, \phi = \text{Const.})$ of the charge density distribution.

interpret the azimuthal current density obtained from the last equation (8.1) as being produced by a rotational movement of the charge density around the z-axis, then it is easy to see, just comparing the units (remember that we chose the units in which $c = 1$), that the induced charge has to "move" with the speed of light. In reality, nothing is moving here: there is just a distribution of static fields $\mathbf{E}$ and $\mathbf{B}$ which produces this illusion, because the Poynting vector $\mathbf{E} \times \mathbf{B}$ has only the azimuthal component. Nevertheless, the illusion produced is the same as for the electron as a whole submitted to the *"zitterbewegung"* with the speed c as it comes out from the relativistic Dirac equation describing the electron.

There is also another striking similarity between the predictions of this model and those of the Dirac equation. Both the lagrangian and the equations it led to (8.1) are invariant with respect to the independent changes of sign, $\mathbf{E} \to -\mathbf{E}$ and $\mathbf{B} \to -\mathbf{B}$.

This means that any static solution generates automatically three other ones, obtained by the inversions of $\mathbf{E}$ and $\mathbf{B}$. Now, the total charge is linear in $\mathbf{E}$, while the total magnetic moment is linear in $\mathbf{B}$; the Poynting vector is proportional to $\mathbf{E} \times \mathbf{B}$, and so will be the total kinetic angular momentum obtained by the integration of $\mathbf{r} \times (\mathbf{E} \times \mathbf{B})$ over the entire space.

The four solutions so obtained can be put together in the following Table 1:

Solution	Energy	Charge	Magnetic μ	Spin
$\mathbf{E},\ \ \mathbf{B}$	m	q	$\boldsymbol{\mu}$	$\mathbf{S}$
$\mathbf{E},\ -\mathbf{B}$	m	q	$-\boldsymbol{\mu}$	$-\mathbf{S}$
$-\mathbf{E},\ \ \mathbf{B}$	m	$-q$	$\boldsymbol{\mu}$	$-\mathbf{S}$
$-\mathbf{E},\ -\mathbf{B}$	m	$-q$	$-\boldsymbol{\mu}$	$\mathbf{S}$

Any static solution is, as a matter of fact, a quadruplet of solutions with the same rest mass. The first two solutions describe a particle with electric charge q and magnetic moment $\boldsymbol{\mu}$ *parallel* to spin $\mathbf{S}$, in states with spin up or down (with respect to the z-axis).

The second pair of solutions describes a particle with the opposite charge $-q$ and magnetic moment *antiparallel* to the spin $\mathbf{S}$, also in two states with spin up or down. This result is identical with the predictions of Dirac's equation for the electron, which leads to the existence of the positron and a half-integer spin, too.

The bad news is that unfortunately a C^∞-class solution of the system (8.1 does not exist. The proof is simple and goes as follows: Knowing that div $\mathbf{B} = 0$, we can write

$$\mathbf{B} \cdot \mathbf{grad}(\mathbf{E} \cdot \mathbf{B}) = \text{div}\ (\mathbf{B}\ (\mathbf{E} \cdot \mathbf{B})) \tag{8.16}$$

Similarly,

$$\mathbf{E} \times \mathbf{grad}(\mathbf{E} \cdot \mathbf{B}) = \mathbf{rot}(\mathbf{E}\ (\mathbf{E} \cdot \mathbf{B})), \tag{8.17}$$

because $\mathbf{rot}\ \mathbf{E} = 0$. This leads to

$$\mathrm{div}\ \left(\mathbf{E} + \frac{3}{e^2}\ \mathbf{B}(\mathbf{E}\cdot\mathbf{B})\right) = 0, \quad \mathbf{rot}\ \left(\mathbf{B} - \frac{3}{e^2}\ \mathbf{E}(\mathbf{E}\cdot\mathbf{B})\right) = 0. \tag{8.18}$$

If the space we are working in has the topology of R^3, and all the functions are supposed to be C^∞-smooth, then the Poincaré lemma states that

$$\mathbf{E} + \frac{3}{e^2}\ \mathbf{B}(\mathbf{E}\cdot\mathbf{B}) = \mathbf{rot}\ \mathbf{C}\ \ ;\ \ \text{and}\quad \mathbf{B} - \frac{3}{e^2}\ \mathbf{E}(\mathbf{E}\cdot\mathbf{B}) = \mathbf{grad}\psi. \tag{8.19}$$

with $\mathbf{C}(\mathbf{r})$ and $\psi(\mathbf{r})$ supposed to be C^∞ smooth (vector and scalar, respectively) functions of $\mathbf{r}$.

Taking the scalar product of the first equation in (8.19) by $\mathbf{E}$ and of the second equation by $\mathbf{B}$ we get (supposing that $\mathbf{E} = -\mathbf{grad}V$):

$$\mathbf{E}^2 + \frac{3}{e^2}\ (\mathbf{E}\cdot\mathbf{B})^2 = \mathbf{E}\cdot\mathbf{rotC} = -(\mathbf{grad}V)\cdot\mathbf{rotC} = -\mathrm{div}\ (V\mathbf{rotC}), \tag{8.20}$$

and

$$\mathbf{B}^2 - \frac{3}{e^2}\ (\mathbf{E}\cdot\mathbf{B})^2 = \mathbf{B}\cdot\mathbf{grad}\psi = \mathrm{div}(\psi\ \mathbf{B}). \tag{8.21}$$

Combining equations (8.20) and (8.21) together, we have

$$\mathbf{E}^2 + \mathbf{B}^2 = \mathrm{div}(\psi\mathbf{B} - V\ \mathbf{rot}\ \mathbf{C}). \tag{8.22}$$

If we want the total energy, as well as the total charge, to be finite, then both $\mathbf{E}$ and $\mathbf{B}$ must decrease at infinity at least as R^{-2}, so that the right-hand side of (8.22) must be of the order of R^{-4}, which means in turn that the vector field $\psi\ \mathbf{B} - V\ \mathbf{rot}\ \mathbf{C}$ is decreasing at infinity as R^{-3}. Applying the Gauss-Ostrogradsky theorem to a finite 3-volume Ω and its 2-dimensional boundary $\partial\Omega$:

$$\int_\Omega \mathrm{div}(\psi\ \mathbf{B} - V\ \mathbf{rot}\ \mathbf{C})\ d^3\mathbf{r} = \int_{\partial\Omega} (\psi\ \mathbf{B} - V\ \mathbf{rot}\ \mathbf{C})\cdot \mathrm{d}\mathbf{\Sigma}, \tag{8.23}$$

we see that the integral of $\mathbf{E}^2 + \mathbf{B}^2$ over a spherical volume of radius R behaves as R^{-1}, i.e. it vanishes when taken over the whole space. Both expressions $\mathbf{E}^2$ and $\mathbf{B}^2$ being positive, this means that $\mathbf{E} = 0$ and $\mathbf{B} = 0$, unless the solution is not C^∞ and the Poincaré lemma does not hold at least on some line or surface.

The impossibility of obtaining a C^∞ solution with finite energy can be also seen if we try to construct it by applying the method of successive approximations in toroidal coordinates.

Now the problem can be reduced down to two equations for two unknown functions, the azimuthal component of the vector potential A_ϕ and the scalar potential V. We can believe that in basic state the dependence on the azimuthal angle ϕ is trivial, therefore we may set

$$A_\phi = A_\phi(\mu,\eta) \quad \text{and} \quad V = V(\mu,\eta) \tag{8.24}$$

The dependence of both potentials on the toroidal angle η must be of the form $\sin(k\eta)$ or $\cos(k\eta)$, $k = 1, 2, ...$; using the substitution

$$A_\phi = u(\eta)\ \sqrt{\cosh\mu - \cos\eta} = (\cosh\mu - \cos\eta)\ G(\mu) \tag{8.25}$$

we make the μ-component of the magnetic field vanish, $B_\mu = 0$.

Along with another substitution

$$V = v(\eta)\ \sqrt{\cosh\mu - \cos\eta} \tag{8.26}$$

the laplacians appearing on the left-hand side of equations (8.1) will have their variables separated. For example, the equation

$$\mathrm{div}\ \mathbf{E} = -\frac{3}{e^2}\ \mathbf{B}\cdot\mathbf{grad}(\mathbf{E}\cdot\mathbf{B}) \tag{8.27}$$

will take on the form:

$$\frac{1}{\sinh\mu}\frac{\partial}{\partial\mu}\left(\sinh\mu\frac{\partial v}{\partial\mu}\right)+\frac{\partial^2 v}{\partial\eta^2}+\frac{1}{4}v=$$

$$\frac{3}{a^2e^2}\frac{(\cosh\mu-\cos\eta)}{\sinh^2\mu}\left[\frac{\partial}{\partial\mu}\left(\sinh\mu\, G(\mu)\right)\right]^2[W(\mu,\eta)], \tag{8.28}$$

with

$$W=[\cosh\mu-\cos\eta)\frac{\partial^2 v}{\partial\eta^2}+4\sin\eta\frac{\partial v}{\partial\eta}+\frac{(5\sin^2\eta+2\cosh\mu\cos\eta-\cos^2\eta)}{4(\cosh\mu-\cos\eta)}. \tag{8.29}$$

Similarly, the laplacian of the function $u(\mu,\eta)$ is equal to some non-linear terms mutiplied by $3/(a^2e^2)$. Developing functions u and v as e.g.

$$\sum_{n=1}^{\infty}[\overset{(1)}{v}_n(\mu)\sin(n\eta)+\overset{(2)}{v}_n(\mu)\cos(n\eta)]$$

the second derivatives in (8.28) will be replaced by n^2v, and the solutions of the *homogeneous equations*,nwhich correspond to the zeroth approximation ($\frac{2}{a^2e^2}=0$) are given as a series in spherical harmonics of half-integer order (cf. Morse and Feshbach).

$$P_{n+\frac{1}{2}}(\cosh\mu) \quad \text{and} \quad Q_{n-\frac{1}{2}}(\cosh\mu) \tag{8.30}$$

The functions $P_{n+\frac{1}{2}}$ display a logarithmic singularity for $\mu=\infty$, i.e. on the circle $\rho=a$, whereas the functions $Q_{n-\frac{1}{2}}$ have a logarithmic singularity for $\mu=0$ (i.e. $\rho,z\to\infty$).

In order to avoid singularity we may use the combination of both, but the price to pay is a discontinuity for some value of μ (on some toroidal surface). If we feed in such a solution to the right-hand side and use the Green functions in order to compute the first correction, we shall be faced with exactly the same problem, because any Green function has at least one singularity of the same kind.

The failure of producing a non-singular soliton is probably due to the fact that we have projected everything onto three space dimensions, discarding the fifth circular one. It seems possible to obtain solitons using the fifth dimension in a non-trivial way, like in the case of Kaluza-Klein monopoles of Sorkin and Gross and Perry.

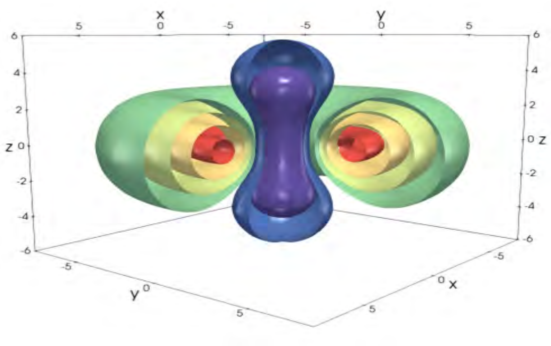

Рис. 6. The constant energy density surfaces in cartesian coordinates

Another development should include the non-abelian generalization of the Kaluza-Klein theory into more dimensions, in which also higher order invariants of the Riemann tensor might be included to the generalized lagrangian.

Recently toroidal solutions for the Higgs-'t Hooft $SU(2)\times U(1)$ monopole were produced numerically by M.S. Volkov *et al.*.

The constant energy density surfaces are represented in cartesian coordinates Fig. 6.

Список литературы/References

1. Nordström, Gunnar. Über die Möglichkeit, das elektromagnetische Feld und das Gravitationsfeld zu vereinigen. *Physikalische Zeitschrift* 15 (1914): 504-506.
2. Kaluza, Th. Zum unitätsproblem der physik. *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)* 1921.arXiv: 1803.08616 (1921): 966-972.
3. Klein, O. Quantentheorie und fünfdimensionale Relativitätstheorie. *Zeitschrift für Physik* 37.12 (1926): 895-906.
4. Einstein, A. *Zu Kaluzas Theorie des Zusammenhanges von Gravitation und Elektrizität: erste-[zweite] Mitteilung.* Verlag der Akademie der Wissenschaften, in Kommission bei Walter de Gruyter u. Company, 1927.
5. Thiry, Y. Les équations de la théorie unitaire de Kaluza. *Comptes Rendus Acad. Sci.(Paris)* 226 (1948): 216.
6. M. Y. Thiry (1948) Sur la régularité des champs gravitationnel et électromagnétique dans les théories unitaires. *Compt. Rend. Acad. Sci. Paris* (in French). 226 pp.1881-1882.
7. J. Pascual. Fünfdimensionale kosmologie. *Astronomische Nachrichten* 276.5-6 (1948): 193-208.
8. Scherrer, W. Uber den Einfluss des metrischen Feldes auf ein skalares Materiefeld. *Helvetica Physica Acta* 22.5 (1949): 537-551.
9. Brans C., and Robert H. Dicke. Mach's principle and a relativistic theory of gravitation. *Physical review* 124.3 (1961): 925.
10. Lovelock D. The Einstein tensor and its generalizations. *Journal of Mathematical Physics* 12.3 (1971): 498-501.
11. Witten E. Search for a realistic Kaluza-Klein theory. *Nuclear Physics B* 186.3 (1981): 412-428.
12. Appelquist Th, Alan C. and Peter George Oliver Freund. *Modern Kaluza-Klein Theories.* (1987).
13. Duff M. J. Kaluza-Klein theory in perspective. *Proc. of the Symposium: The Oskar Klein Centenary, World Scientific, Singapore.* 1994.
14. Overduin J. M., and Paul S. W. Kaluza-klein gravity. *Physics reports* 283.5-6 (1997): 303-378.
15. Wesson P. S. *Five-dimensional physics: classical and quantum consequences of Kaluza-Klein cosmology.* World Scientific, 2006.
16. Coquereaux R. and Gilles E.F. The theory of Kaluza-Klein-Jordan-Thiry revisited. *Annales de l'IHP Physique théorique.* Vol. 52. No. 2. 1990.
17. Kerner R. Geometrical background for the unified field theories: the Einstein-Cartan theory over a principal fibre bundle. *Annales de l'institut Henri Poincaré. Section A, Physique Théorique.* Vol. 34. No. 4. 1981.
18. Kerner R. Multiple fiber bundles and gauge theories of higher order. *Journal of Mathematical Physics* 24.2 (1983): 356-360.
19. Kerner R. Electrodynamique non linéaire en théorie de Kaluza et Klein. *Comptes rendus de l'Académie des sciences. Série 2, Mécanique, Physique, Chimie, Sciences de l'univers, Sciences de la Terre* 304.12 (1987): 621-624.
20. Kerner R. Non-linear electrodynamics derived from the Kaluza-Klein theory. *arXiv preprint arXiv:2303.10603* (2023).
21. Bruno G., and Kerner R. Cosmology in ten dimensions with the generalised gravitational Lagrangian. *Classical and Quantum Gravity* 5.2 (1988): 339.
22. Ghosh S. G., and Sunil D. M. Cloud of strings for radiating black holes in Lovelock gravity. *Physical Review D* 89.8 (2014): 084027.
23. Matzner R. A. and Mezzacappa A. Three-dimensional closed universes without collapse in a five-

dimensional Kaluza-Klein theory. *Physical Review D* 32.12 (1985): 3114.

24. Copeland, Edmund J., and David J. Toms. Stability of self-consistent higher-dimensional cosmological solutions. *Physical Review D* 32.8 (1985): 1921.
25. Chodos, Alan, and Steven Detweiler. Where has the fifth dimension gone?. *Physical Review D* 21.8 (1980): 2167.
26. Sahdev, Deshdeep. Perfect-fluid higher-dimensional cosmologies. *Physical Review D* 30.12 (1984): 2495.
27. Srivastava, S. K. Some aspects of Kaluza-Klein cosmology. *Pramana* 49 (1997): 323-370.
28. Sorkin, Rafael D. Kaluza-klein monopole. *Physical Review Letters* 51.2 (1983): 87.
29. Gross, David J., and Malcolm J. Perry. Magnetic monopoles in Kaluza-Klein theories. *Nuclear Physics B* 226.1 (1983): 29-48.
30. Gervalle, Romain, and Mikhail S. Volkov. Electroweak multi-monopoles. *Nuclear Physics B* 987 (2023): 116112.

Авторы

Ричард Кернер, д.ф.-м.н., профессор, Сорбонский университет, пл. Жюсьё, d.4, г. Париж, 75005, Франция.
E-mail: richard.kerner@sorbonne-universite.fr

Просьба ссылаться на эту статью следующим образом:

Кернер Р. Обобщенные модели Калуцы - Клейна с лагранжианами Гаусса - Бонне. *Пространство, время и фундаментальные взаимодействия.* 2023. № 3-4. С. 166–187.

Authors

Kerner Richard, Ph.D., Professor, LPTMC, Sorbonne Universite - CNRS UMR 7600, Place Jussieu, 4, Paris, 75005, France.
E-mail: richard.kerner@sorbonne-universite.fr

Please cite this article in English as:

Kerner R. Generalized Kaluza-Klein models with Gauss-Bonnet lagrangians. *Space, Time and Fundamental Interactions*, 2023, no. 3-4, pp. 166–187.