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УРАВНЕНИЕ ДИРАКА И ФЕРМИОННАЯ АЛГЕБРА

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В данной работе рассматривается структура уравнения Дирака и дается новая трактовка уравнения Дирака в пространстве 1+1.

Ключевые слова: Алгебра Клиффорда, уравнение Дирака, доска Фейнмана.

THE DIRAC EQUATION AND A FERMIONIC ALGEBRA

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This paper examines the structure of the Dirac equation and gives a new treatment of the Dirac equation in 1+1 spacetime.

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1. Introduction

This paper is a discussion of the structure of the Dirac equation, primarily in the case of one dimension of space and one dimension of time ($1+1$ spacetime). We reformulate the Dirac operator $\mathcal{D}$ so that there is a nilpotent element U, with $U^2 = 0$, in the Clifford algebra such that for a plane wave ψ, $\mathcal{D}\psi = U\psi$. This means that $U\psi$ is a solution to the Dirac equation since $\mathcal{D}(U\psi) = U^2\psi = 0 \times \psi = 0$. We explain this formulation in Section 2 of the paper, and use it in Section 3 to reformulate a nilpotent version of the Dirac equation for (1+1) spacetime in light cone coordinates. We can then give a solution to the Dirac equation by the method just indicated and we can compare this solution with the solutions already understood in relation to the Feynman checkerboard model. In the course of this reformulation we see that the transition to light cone coordinates corresponds to a rewriting of the Clifford algebra for the Dirac equation to a Fermionic algebra linked with a Clifford algebra. We obtain the following result (in summary).

We have the (1+1) Dirac equation in light cone coordinates (l, r), using the light cone Dirac operator

$$\mathcal{D} = A\partial/\partial l + B\partial/\partial r - \alpha m.$$

The elements A, B, α satisfy the algebra relations:

$$AB + BA = 1, AB - BA = \alpha, A^2 = B^2 = 0, \alpha^2 = 1,$$

$$A\alpha = -A, \alpha A = A, B\alpha = B, \alpha B = -B.$$

Note that in this algebra the elements A and B form a Fermion algebra, each squaring to 0 and satisfying $AB + BA = 1$. The element α has square one, and can be regarded as a Clifford algebra element

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interacting with A and B. This *special Fermion algebra* is the key to the calculations in this paper and we will study it further in subsequent work.

The rest of section 3 is a discussion of the relationship of our results in the paper with the Feynman checkerboard model and with our previous work on that model [1, 2].

The appendix discusses how the nilpotent and Majorana operators arise in three dimensions of space and one dimension of time. This appendix provides a link between our work and the work of Peter Rowlands [10]. The Majorana Dirac equation can be written as follows:

$$(\partial/\partial t + \hat{\eta}\eta\partial/\partial x + \epsilon\partial/\partial y + \hat{\epsilon}\eta\partial/\partial z - \hat{\epsilon}\hat{\eta}\eta m)\psi = 0$$

where η and ϵ are the generators of a Clifford algebra with $\eta^2 = \epsilon^2 = 1$ and $\eta\epsilon + \epsilon\eta = 0$, and $\hat{\epsilon}, \hat{\eta}$ form a copy of this algebra that commutes with it. This combination of a Clifford algebra with itself is the underlying structure of Majorana Fermions. In the appendix we apply our methods to the Majorana Dirac Equation and give actual real solutions to the equation. These solutions make direct use of the Majorana Fermion Clifford algebra. This shows explicitly that Fermions and Majorana Fermions are related by the algebraic transformation between Fermion and Clifford algebra.

Remark. The more intricate algebra in this paper such as the special Fermion algebra described above can be regarded as coming from the patterns of the split quaternions seen as the Clifford algebra with generators α, β and relations $\alpha^2 = \beta^2 = 1, \alpha\beta + \beta\alpha = 0$. From these relations it follows that $(\alpha\beta)^2 = -1$ and if we write

$$U = \alpha\beta E + \alpha p + \beta m$$

where E, p, m are scalars commuting with the algebra elements, then

$$U^2 = -E^2 + p^2 + m^2$$

since the cross terms all vanish in the product. Thus when $E^2 = p^2 + m^2$ we have a non-trivial nilpotent element U in the Clifford algebra with $U^2 = 0$. This is the beginning of the key relationship of nilpotent algebra elements and Fermions as it occurs in the work of Peter Rowlands [10] and it is the keystone of the work in this paper as well.

2. The Dirac Equation

We begin by recalling how Dirac constructed his equation. By convention we take the speed of light to be equal to 1. Then energy E, momentum p and mass m are related through special relativity by the equation

$$E^2 = p^2 + m^2.$$

Dirac looked for an algebraic square root of $p^2 + m^2$ so that he could have a linear operator corresponding to E that would take the same role as the Hamiltonian in the Schrödinger equation.

We first take the case of one dimension of space and one dimension of time so that p is a scalar. The quantum operator for momentum is

$$\hat{p} = -i\partial/\partial x,$$

the operator for energy is

$$\hat{E} = i\partial/\partial t,$$

and the operator for mass is

$$\hat{m} = m.$$

We can write an operator equation

$$\hat{E} = \alpha\hat{p} + \beta\hat{m},$$

where α and β are elements of a a possibly non-commutative, associative algebra.

Then

$$\hat{E}^2 = \alpha^2\hat{p}^2 + \beta^2\hat{m}^2 + \hat{p}\hat{m}(\alpha\beta + \beta\alpha).$$

Hence we have $\hat{E}^2 = \hat{p}^2 + \hat{m}^2$ if we take

$$\alpha^2 = \beta^2 = 1,$$

$$\alpha\beta + \beta\alpha = 0.$$

The algebra so generated by α and β is a simplest Clifford algebra.

Remark. Note that the Clifford algebra with generators α, β with relations as given above is often called the *split quaternions*. If we introduce a commuting (with α and β) square root of minus one, denoted i with $i^2 = -1$, and let $I = i\alpha, J = i\beta, K = \beta\alpha$, then it is the case that $I^2 = J^2 = K^2 = IJK = -1$ and thus we obtain the quaternions from the split quaternions.

Remark. In general we take a Clifford algebra to be an associative algebra with abstract generators $e_1, e_2, \cdots, e_n$ so that $e_k^2 = 1$ for all k and $e_r e_s + e_s e_r = 0$ whenever $r \neq s$. The generators are usually taken to be an orthonormal basis for a vector space over a field.

Remark. Clifford algebras and Fermion algebras are related to one another by a transformation that we illustrate here for the split quaternions. Let

$$U = (\alpha + i\beta)/2,$$

$$V = (\alpha - i\beta)/2.$$

then

$$U^2 = V^2 = (\alpha^2 - \beta^2 \pm i(\alpha\beta + \beta\alpha))/4 = 0,$$

$$UV + VU = (U + V)^2 = \alpha^2 = 1.$$

The relations $U^2 = V^2 = 0$ and $UV + VU = 1$ are characteristic of Fermion algebra and correspond to properties of creation and annihilation operators for Fermions. We will see that Fermion algebras arise naturally in relation to Clifford algebra formulations for the Dirac equation.

The Dirac equation is the operator equation

$$\hat{E}\psi = \alpha\hat{p}\psi + \beta\hat{m}\psi.$$

Thus the Dirac equation is the differential equation below.

$$i\partial\psi/\partial t = -i\alpha\partial\psi/\partial x + \beta m\psi.$$

We begin by discussing this version of the Dirac equation in $1+1$ spacetime, constructing solutions via light cone reformulaiton and we discuss the Feynman checkerboard model. In the Appendix, we explain how to extend these formulations to $(3 + 1)$ spacetime.

2.1. The Nilpotent Reformulation of the Dirac Equation

We can define the *Dirac operator* $\mathcal{O}$ as follows: Let $\mathcal{O} = i\partial/\partial t + i\alpha\partial/\partial x - \beta m$. Then the Dirac equation takes the form $\mathcal{O}\psi(x,t) = 0$.

Note that $\mathcal{O}e^{i(px-Et)} = (E - \alpha p - \beta m)e^{i(px-Et)}$.

We let $\Delta = (E - \alpha p - \beta m)$ and let

$$U = \alpha\beta\Delta = \alpha\beta E + \beta p - \alpha m,$$

so that

$$U^2 = -E^2 + p^2 + m^2 = 0.$$

(Note that $(\alpha\beta)^2 = \alpha\beta\alpha\beta = -\alpha\alpha\beta\beta = -1$ and that the cross terms cancel.)

Remark. It is of interest to note that in the split quaternions we have elements of the form

$$U = \alpha\beta E + \beta p - \alpha m$$

such that $U^2 = -E^2 + p^2 + m^2$ so that U is nilpotent of order two exactly when $E^2 = p^2 + m^2$.
We now multiply the operator $\mathcal{O}$ by $\alpha\beta$ on the left, obtaining the operator

$$\mathcal{D} = \alpha\beta\mathcal{O} = i\alpha\beta\partial/\partial t + i\beta\partial/\partial x - \alpha m.$$

The Dirac equation is equivalent to the equation $\mathcal{D}\psi = 0$. Furthermore, we have have $\mathcal{D}(e^{i(px-Et)}) = Ue^{i(px-Et)}$. Thus for $\psi = e^{i(px-Et)}$, we have $\mathcal{D}(\psi) = U\psi$ and $\mathcal{D}(U\psi) = U^2\psi = 0$. Thus U acts as a creation operator producing a solution to the Dirac equation.

This idea for reconfiguring the Dirac equation in relation to nilpotent algebra elements U is due to Peter Rowlands [10]. Rowlands does this in the context of quaternion algebra. The solution to the Dirac equation that we have found is expressed in Clifford algebra. It can be articulated into specific vector solutions by using a matrix representation of the algebra.

2.2. Fermion Operators

We see that $U = \alpha\beta E + \beta p - \alpha m$ with $U^2 = 0$ is the essence of this plane wave solution to the Dirac equation. It is natural to compare this algebra structure with algebra of creation and annihilation operators that occur in quantum field theory.
If we let $\tilde{\psi} = e^{i(px+Et)}$ (reversing time), then we have $\mathcal{D}\tilde{\psi} = (\beta\alpha E + \beta p - \alpha m)\psi = U^\dagger\tilde{\psi}$, giving a definition of $U^\dagger$ corresponding to the anti-particle for $U\psi$.
We have $U = \alpha\beta E + \beta p - \alpha m$ and $U^\dagger = \beta\alpha E + \beta p - \alpha m$.
Note that here we have

$$(U + U^\dagger)^2 = (2\beta p + \alpha m)^2 = 4(p^2 + m^2) = 4E^2,$$

and

$$(U - U^\dagger)^2 = -(2\alpha\beta E)^2 = -4E^2.$$

We have that $U^2 = (U^\dagger)^2 = 0$ and $UU^\dagger + U^\dagger U = 4E^2$. Thus we have a direct appearance of the Fermion algebra corresponding to the Fermion plane wave solutions to the Dirac equation.

Normalizing by dividing by $2E$ we have $A = (\beta p - \alpha m)/E$ and $B = i\beta\alpha$. so that $A^2 = B^2 = 1$ and $AB + BA = 0$. then $U = (A + Bi)E$ and $U^\dagger = (A - Bi)E$, showing how the Fermion operators are expressed in terms of the simpler Clifford algebra of Majorana operators (A and B generating the split quaternions).

The decomposition of $U$and $U^\dagger$ into the corresponding Majorana Fermion operators with $A^2 = 1$ is exactly equivalent to $E^2 = p^2 + m^2$.

3. Spacetime in 1+1 Dimensions

We begin this section by discussing an algebra that is directly related to Clifford algebra. As we shall see, this algebra is also inherent in the Dirac equation when we use light cone coordinates.

3.1. Clifford algebra and Fermion algebra.

Suppose that we have a Clifford algebra generated by elements ϵ and η with $\epsilon^2 = \eta^2 = 1$ and $\epsilon\eta + \eta\epsilon = 0$. Then we can define new elements a and b by the equations

$$\eta = a + b,$$

$$\epsilon\eta = a - b.$$

This means that

$$a = \frac{1}{2}(1+\epsilon)\eta,$$

$$b = \frac{1}{2}(1-\epsilon)\eta,$$

from which it follows that

$$a^2 = b^2 = 0, ab + ba = 1.$$

Note that we are given that the starting Clifford algebra is associative and so further identities such as

$$aba = a, bab = b, abab = ab, baba = ba$$

follow easily from the given identities. We call an associative algebra generated by a, b with

$$a^2 = b^2 = 0, ab + ba = 1$$

a *Fermion algebra* since the annihilation, creation algebra for Fermions in quantum theory satisfies these identities. We see here that Clifford algebras (with an even number of generators) and Fermion algebras are interchangeable via the above transformations. This fact has been used by writers on Clifford algebras, [11] since it is useful to have projector properties such as $(ab)(ab) = ab$.

Example. In two by two matrix algebra, we can take

$$\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = a + b.$$

Here

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Thus

$$ab = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, ba = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

so that

$$a^2 = b^2 = 0,$$

$$a + b = \eta,$$

$$a - b = \epsilon\eta,$$

$$ab + ba = 1,$$

$$ab - ba = \epsilon.$$

Remark. The above construction of Fermion algebra from Clifford algebra occurs without invoking an extra commuting square root of negative unity. It is common in physical applications to use a parallel construction involving i where $i^2 = -1$ and i commutes with all elements of the algebra. One can then define $\psi = \frac{1}{2}(\eta + i\epsilon)$ and $\psi^\dagger = \frac{1}{2}(\eta - i\epsilon)$. It follows that $\psi^2 = (\psi^\dagger)^2 = 0$ and $\psi\psi^\dagger + \psi^\dagger\psi = 1$, and one has a Fermion algebra with complex conjugation constructed in relation to a Clifford algebra. Another relation with a commuting i occurs if we take

$$a = (i/2)(\alpha\beta + \beta)$$

$$b = (i/2)(\alpha\beta - \beta)$$

where α and β form a Clifford algebra with $\alpha^2 = \beta^2 = 1$ and $\alpha\beta + \beta\alpha = 0$. Then a and b satisfy the Fermion relations and

$$ab + ba = 1,$$

$$ab - ba = \alpha,$$

but

$$a + b = i\alpha\beta,$$

$$a - b = i\beta.$$

Notice that $(i\alpha\beta)^2 = +1$ while $(i\beta)^2 = -1$. Thus we can regard this as a re-writing of the previous pattern with

$$i\alpha\beta = \eta$$

and

$$i\beta = \epsilon\eta$$

so that

$$\alpha = \beta\beta\alpha = -\beta\alpha\beta = i\beta[\alpha\beta = \epsilon\eta\eta = \epsilon.$$

This means that this Fermion algebra can occur with or without the explicit commuting square root of negative unity, i.

3.2. The Dirac Equation in Light Cone Coordinates

Light cone coordinates r and l are defined by

$$r = (x + t)/2$$

and

$$l = (x - t)/2.$$

Note that $4rl = x^2 - t^2$. Thus the light cone in (x, t) Minkowski space (light speed $c = 1$) is descibed by the equations $r = 0$ or $l = 0$.

Recall the translation of operators to light cone coordinate operators.

$$\hat{E} = i\partial/\partial t = (i/2)(\partial/\partial r + \partial/\partial l)$$

$$\hat{p} = (1/i)\partial/\partial x = (1/2i)(\partial/\partial r - \partial/\partial l)$$

Here is the nilpotent version of the Dirac operator as we have formulated it.

$$\mathcal{D} = \alpha\beta\hat{E} + \beta\hat{p} - \alpha\hat{m}$$

We translate this operator into light cone coordinates.

$$\mathcal{D} = \alpha\beta((i/2)(\partial/\partial r + \partial/\partial l)) + \beta((1/2i)(\partial/\partial r - \partial/\partial l)) - \alpha m$$

$$\mathcal{D} = i[(\alpha\beta + \beta)/2]\partial/\partial l + i[(\alpha\beta - \beta)/2]\partial/\partial r - \alpha m$$

Thus

$$\mathcal{D} = A\partial/\partial l + B\partial/\partial r - \alpha m$$

$$A = (i/2)(\alpha\beta + \beta)$$

$$B = (i/2)(\alpha\beta - \beta)$$

As the reader can see, we arrive at algebraic coefficients that we have described above as the Fermion algebra associated with the Clifford algebra generated by α and β.

$$A + B = i\alpha\beta$$

$$A - B = i\beta$$

Further relations take the form:

$$AB + BA = 1, AB - BA = \alpha, A^2 = B^2 = 0, \alpha^2 = 1$$

$$A\alpha = -A, \alpha A = A, B\alpha = B, \alpha B = -B.$$

Thus

$$A\alpha + \alpha A = 0, B\alpha + \alpha B = 0$$

$$A\beta + \beta A = i, B\beta + \beta B = -i$$

3.3. Plane waves in light cone coordinates.

Let

$$\psi = e^{i(rX - lY)}$$

where

$$X = p - E$$

and

$$Y = p + E.$$

Note that $XY = -m^2$. This is the plane wave rewritten in light cone coordinates. Then with

$$\mathcal{D} = A\partial/\partial l + B\partial/\partial r - \alpha m,$$

$$\mathcal{D}\psi = U\psi$$

where

$$U = -iAX + iBY - \alpha m.$$

Thus

$$U^2 = ABXY + BAXY + m^2 = XY + m^2 = p^2 - E^2 + m^2 = 0.$$

Note that with

$$U^\dagger = -iAY + iBX - \alpha m$$

we have

$$(U^\dagger)^2 = 0$$

and

$$UU^\dagger + U^\dagger U = 4E^2$$

Summary. We have the (1+1) Dirac equation in light cone coordinates, using the light cone Dirac operator

$$\mathcal{D} = A\partial/\partial l + B\partial/\partial r - \alpha m.$$

The elements A, B, α satisfy the Fermionic algebra relations:

$$AB + BA = 1, AB - BA = \alpha, A^2 = B^2 = 0, \alpha^2 = 1,$$

$$A\alpha = -A, \alpha A = A, B\alpha = B, \alpha B = -B.$$

We can directly see the action of the light cone Dirac operator on a plane wave expressed in light cone coordinates. The plane wave is given by the formula

$$\psi = e^{i(rX - lY)}$$

where $X = p - E$ and $Y = p + E$. Thus $XY = -m^2$. Then $\mathcal{D}\psi = U\psi$ where $U = -iAX + iBY - \alpha m$, and $U^2 = 0$.

Note that with $U^\dagger = -iAY + iBX - \alpha m$ we have $(U^\dagger)^2 = 0$ and $UU^\dagger + U^\dagger U = 4E^2$. Thus we have rewritten the nilpotent Dirac operator and its equation directly in light cone coordinates, with the help of the Fermionic algebra.

3.4. Solving the 1+1 Dirac Equation

The (real valued) Majorana version of the Dirac operator

$$\mathcal{D} = A\partial/\partial l + B\partial/\partial r - \alpha m$$

that we have discussed above can be taken with the representation

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \alpha = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, AB - BA = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \alpha.$$

Letting $\Theta = rX - lY$, and $S = Sin(\Theta), C = Cos(\Theta)$, we have

$$U\psi = U(C + iS) = (AXS - BYS - \alpha mC) + i(-AXC + BYC - \alpha mS).$$

In the matrix representation we find

$$AXS - BYS - \alpha mC = \begin{pmatrix} mC & -YS \\ XS & -mC \end{pmatrix}.$$

And from this, letting

$$\psi_1 = mC, \psi_2 = XS$$

we have

$$\partial\psi_1/\partial r = -mXS = -m\psi_2$$

and

$$\partial\psi_2/\partial l = -XYC = m^2C = m\psi_1$$

Thus

$$\partial\psi_1/\partial r = -m\psi_2$$

$$\partial\psi_2/\partial l = m\psi_1$$

Note that these equations are satisfied by

$$\psi_1 = -mSin(-(E-p)r - (E+p)l),$$

$$\psi_2 = (E+p)Cos(-(E-p)r - (E+p)l)$$

exactly when $E^2 = p^2 + m^2$ as we have assumed. It is quite interesting to see these direct solutions to the Dirac equation emerge in this $1+1$ case. The solutions are fundamental and they are distinct from the usual solutions that emerge from the Feynman checkerboard model [1, 2]. It is the above equations that form the basis for the Feynman checkerboard model that is obtained by examining paths in a discrete Minkowski plane generating a path integral for the Dirac equation.

Remark. Note that a simplest instance of the above form of solution is obtained by writing

$$e^{i(r+l)} = cos(r+l) + isin(r+l) = \sum_{n=0}^{\infty}(\sqrt{-1})^n \sum_{i+j=n} \frac{r^i}{i!}\frac{l^j}{j!}.$$

Then with $\psi_2 = cos(r+l)$ and $\psi_1 = sin(r+l)$ we have $\partial\psi_1/\partial l = \psi_2$, $\partial\psi_2/\partial r = -\psi_1$, solving the Dirac equation in the case where $m = 1$.

Remark. Let $\psi_R = \sum_{k=0}^{\infty}(-1)^k \frac{r^{k+1}}{(k+1)!}\frac{l^k}{k!}$, $\psi_L = \sum_{k=0}^{\infty}(-1)^k \frac{r^k}{k!}\frac{l^{k+1}}{(k+1)!}$, $\psi_0 = \sum_{k=0}^{\infty}(-1)^k \frac{r^k}{k!}\frac{l^k}{k!}$. Then $\psi_1 = \psi_0 + \psi_L$ and $\psi_2 = \psi_0 - \psi_R$ give a solution to the Dirac equation in light cone coordinates as we have written it above with $m = 1 : \partial\psi_1/\partial l = \psi_2, \partial\psi_2/\partial r = -\psi_1$. These series are shown in [2] to be a natural limit of evaluations of sums of discrete paths on the Feynman checkerboard. The key to our earlier approach is that if one defines

$$C[\Delta]^x_k = \frac{(x)(x-\Delta)(x-2\Delta)\cdots(x-(k-1)\Delta)}{k!},$$

Then $C[\Delta]^x_k$ takes the role of $\frac{x^k}{k!}$ for discrete different derivatives with step length Δ and it can be interpreted as a choice coefficient. A Feynman path on a rectangle in Minkowski space can be interpreted as two choice of k or $k+1$ points along the r and l edges of the rectangle. Thus the products in the limit expressions of the form $\frac{r^k}{k!}\frac{l^{k+1}}{(k+1)!}$ or $\frac{r^k}{k!}\frac{l^k}{k!}$ correspond to paths on the checkerboard with k corners in a limit where there are infinitely many such paths. The details are in our paper [2]. The solutions we have given above, motivated by the Majorana algebra, are related in form to these path sum solutions. Our solutions contain more information, related to the factorization $(E-p)(E+p) = E^2 - p^2 = m^2$. In the usual checkerboard solution the propagators only know about the mass and not its factorization relative to energy and momentum. More work needs to be done to fully understand the relationship of solutions to the Dirac equation and path summations.

Path Sum Derivation.

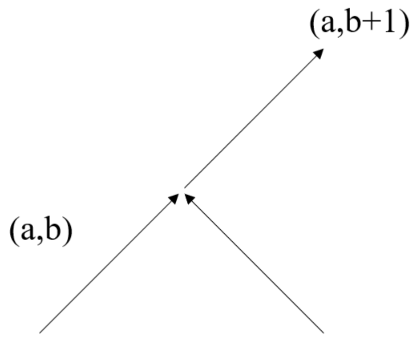

Рис. 1. Path Summation

Here we describe the Feynman checkerboard model where light-speed paths p with corners, in Minkowski space, are each evaluated by $i^{c(p)}$ where $c(p)$ denotes the number of corners in the path. Let (a, b) denote a point in discrete Minkowski spacetime in light cone coordinates. Thus a denotes the number of steps taken to the left and b denotes the number of steps taken to the right. We let $\psi_L(a, b)$ denote the sum over the paths that enter the point (a, b) from the left and $\psi_R(a, b)$ the sum over the paths that enter (a, b) from the right. View Figure 1.

It is clear from the diagram in the figure that

$$\psi_L(a, b+1) = \psi_L(a, b) + i\psi_R(a, b).$$

Thus we have a discrete version of the Dirac equation in light cone coordinates that is satisfied by the Feynman path summation. If we adjust the step sizes and take a limit we find

$$\partial\psi_L/\partial r = i\psi_R$$

and similarly

$$\partial\psi_R/\partial l = i\psi_L.$$

This pair of equations is the the Dirac Equation in light cone coordinates. When we take the the evalutation of a path to be $(-1)^{c(p)}$, we obtain the real version of the Dirac equation, as discussed above.

It remains to be seen how our plane wave solutions of the (1+1) Dirac equation in light cone coordinates are related to the Feynman path summation.

4. Appendix - Writing in the Full Dirac Algebra

We have written the Dirac equation in one dimension of space and one dimension of time. We now boost the formalism directly to three dimensions of space. We take an independent Clifford algebra generated by $\sigma_1, \sigma_2, \sigma_3$ with $\sigma_i^2 = 1$ for $i = 1, 2, 3$ and $\sigma_i\sigma_j = -\sigma_j\sigma_i$ for $i \neq j$. Now assume that α and β as we have used them above generate an independent Clifford algebra that commutes with the algebra of the σ_i. Replace the scalar momentum p by a 3-vector momentum $p = (p_1, p_2, p_3)$ and let $p \bullet \sigma = p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_3$. We replace $\partial/\partial x$ with $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_2)$ and $\partial p/\partial x$ with $\nabla \bullet p$. We then have the following form of the Dirac equation.

$$i\partial\psi/\partial t = -i\alpha\nabla \bullet \sigma\psi + \beta m\psi.$$

Let $\mathcal{O} = i\partial/\partial t + i\alpha\nabla \bullet \sigma - \beta m$ so that the Dirac equation takes the form $\mathcal{O}\psi(x,t) = 0$.
In analogy to our previous discussion we let $\psi(x,t) = e^{i(p \bullet x - Et)}$ and construct solutions by first applying the Dirac operator to this ψ. The two Clifford algebras interact to generalize directly the nilpotent solutions and Fermion algebra,that we have detailed for one spatial dimension, to this three dimensional case. To this purpose the modified Dirac operator is

$$\mathcal{D} = i\alpha\beta\partial/\partial t + \beta\nabla \bullet \sigma - \alpha m.$$

And we have that $\mathcal{D}\psi = U\psi$ where $U = \alpha\beta E + \beta p \bullet \sigma - \alpha m$. We have that $U^2 = 0$ and $U\psi$ is a solution to the modified Dirac Equation, just as before. And just as before, we can articulate the structure of the Fermion operators and locate the corresponding Majorana Fermion operators.

4.1. Majorana Fermions

There is more to do. We now discuss making Dirac algebra distinct from the one generated by $\alpha, \beta, \sigma_1, \sigma_2, \sigma_3$ to obtain an equation that can have real solutions. This was the strategy that Majorana [3] followed to construct his Majorana Fermions. A real equation can have solutions that are invariant under complex conjugation and so can correspond to particles that are their own anti-particles. We will describe this Majorana algebra in terms of the split quaternions ϵ and η. For convenience we use the matrix representation given below.

$$\epsilon = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\hat{\epsilon}$ and $\hat{\eta}$ generate another, independent algebra of split quaternions, commuting with the first algebra generated by ϵ and η. Then a totally real Majorana Dirac equation can be written as follows:

$$(\partial/\partial t + \hat{\eta}\eta\partial/\partial x + \epsilon\partial/\partial y + \hat{\epsilon}\eta\partial/\partial z - \hat{\epsilon}\hat{\eta}\eta m)\psi = 0.$$

To see that this is a correct Dirac equation, note that

$$\hat{E} = \alpha_x\hat{p_x} + \alpha_y\hat{p_y} + \alpha_z\hat{p_z} + \beta m$$

(Here the "hats" denote the quantum differential operators corresponding to the energy and momentum.) will satisfy

$$\hat{E}^2 = \hat{p_x}^2 + \hat{p_y}^2 + \hat{p_z}^2 + m^2$$

if the algebra generated by $\alpha_x, \alpha_y, \alpha_z, \beta$ has each generator of square one and each distinct pair of generators anti-commuting. From there we obtain the general Dirac equation by replacing $\hat{E}$ by $i\partial/\partial t$, and $\hat{p_x}$ with $-i\partial/\partial x$ (and same for y, z).

$$(i\partial/\partial t + i\alpha_x\partial/\partial x + i\alpha_y\partial/\partial y + i\alpha_z\partial/\partial y - \beta m)\psi = 0.$$

This is equivalent to

$$(\partial/\partial t + \alpha_x \partial/\partial x + \alpha_y \partial/\partial y + \alpha_z \partial/\partial y + i\beta m)\psi = 0.$$

Thus, here we take

$$\alpha_x = \hat{\eta}\eta, \alpha_y = \epsilon, \alpha_z = \hat{\epsilon}\eta, \beta = i\hat{\epsilon}\hat{\eta}\eta,$$

and observe that these elements satisfy the requirements for the Dirac algebra. Note how we have a significant interaction between the commuting square root of minus one (i) and the element $\hat{\epsilon}\hat{\eta}$ of square minus one in the split quaternions. This brings us back to considerations about the source of the square root of minus one. Both viewpoints combine in the element $\beta = i\hat{\epsilon}\hat{\eta}\eta$ that makes this Majorana algebra work. Since the algebra appearing in the Majorana Dirac operator is constructed entirely from two commuting copies of the split quaternions, there is no appearance of the complex numbers, and when written out in 2×2 matrices we obtain coupled real differential equations to be solved. This is a beginning of a new study of Majorana Fermions. For more information about this viewpoint, see [9]. In the next section we rewrite the Majorana Dirac operator, guided by nilpotents, obtaining solutions that directly use the Majorana Fermion operators.

4.2. Nilpotents, Majorana Fermions and the Majorana-Dirac Equation

Let $\mathcal{D} = (\partial/\partial t + \hat{\eta}\eta\partial/\partial x + \epsilon\partial/\partial y + \hat{\epsilon}\eta\partial/\partial z - \hat{\epsilon}\hat{\eta}\eta m)$. In the last section we have shown how $\mathcal{D}$ can be taken as the Majorana operator through which we can look for real solutions to the Dirac equation. Letting $\psi(x,t) = e^{i(p\bullet r - Et)}$, we have

$$\mathcal{D}\psi = (-iE + i(\hat{\eta}\eta p_x + \epsilon p_y + \hat{\epsilon}\eta p_z) - \hat{\epsilon}\hat{\eta}\eta m)\psi.$$

Let

$$\Gamma = (-iE + i(\hat{\eta}\eta p_x + \epsilon p_y + \hat{\epsilon}\eta p_z) - \hat{\epsilon}\hat{\eta}\eta m)$$

and

$$U = \epsilon\eta\Gamma = (i(-\eta\epsilon E - \hat{\eta}\epsilon p_x + \eta p_y - \epsilon\hat{\epsilon}p_z) + \epsilon\hat{\epsilon}\hat{\eta}m).$$

The element U is nilpotent, $U^2 = 0$, and we have that $U = A + iB$, $AB + BA = 0$, $A = \epsilon\hat{\epsilon}\hat{\eta}m$, $B = -\eta\epsilon E - \hat{\eta}\epsilon p_x + \eta p_y - \epsilon\hat{\epsilon}p_z$, $A^2 = -m^2$, and $B^2 = -E^2 + p_x^2 + p_y^2 + p_z^2 = -m^2$.
Letting $\nabla = \epsilon\eta\mathcal{D}$, we have a new Majorana Dirac operator with $\nabla\psi = U\psi$ so that $\nabla(U\psi) = U^2\psi = 0$. Letting $\theta = (p \bullet r - Et)$, we have $U\psi = (A + Bi)e^{i\theta} = (A + Bi)(Cos(\theta) + iSin(\theta)) = (ACos(\gamma) - BSin(\theta)) + i(BCos(\theta) + ASin(\theta))$.
Thus we have found two real solutions to the Majorana Dirac Equation:

$$\Phi = ACos(\theta) - BSin(\theta),$$

$$\Psi = BCos(\theta) + ASin(\theta)$$

with $\theta = (p \bullet r - Et)$ and A and B the Majorana operators

$$A = \epsilon\hat{\epsilon}\hat{\eta}m,$$

$$B = -\eta\epsilon E - \hat{\eta}\epsilon p_x + \eta p_y - \epsilon\hat{\epsilon}p_z.$$

Note how the Majorana Fermion algebra generated by A and B comes into play in the construction of these solutions. This answers a natural question about the Majorana Fermion operators. Should one take the Majorana operators themselves seriously as representing physical states? Our calculation suggests that one should take them seriously.

In other work [4, 5, 6, 7] we review the main features of recent applications of the Majorana algebra and its relationships with representations of the braid group and with topological quantum computing. The present analysis of the Majorana Dirac equation first appears in our paper [9].

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