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АЛГЕБРОДИНАМИКА: В ПОИСКАХ ИСТИННОЙ АЛГЕБРАИЧЕСКОЙ “МИРОВОЙ” СТРУКТУРЫ

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Представлены основные принципы т.н. *алгебродинамического* подхода к построению единой теории поля, и его реализация на основе линейной алгебры *комплексных кватернионов*. Далее обсуждаются возможные реализации алгебродинамики на многообразии, оснащенном структурой группы Ли или ее специальными обобщениями – алгебрическими структурами (АС) с *единственной операцией*, заданной *единственным определяющим соотношением* для трех либо четырех элементов (аналогом требования ассоциативности для группы Ли). Заданная таким образом т.н. *инвариантная АС* оказывается эквивалентной группе Ли, однако допускает тем самым неканоническое введение последней с использованием единственного определяющего соотношения. На роль “Мировой” АС предложены и предварительно изучены еще два примечательных их типа, а именно т.н. *автоморфная* и *универсальная* АС. Фундаментальные физические поля $F(x)$ рассматриваются как нетривиальные отображения элементов АС отвечающие, в частности, умножению элемента “на себя”, $F(x) = x \cdot x$.

Ключевые слова: Бикватернионы, группы Ли, инвариантная алгебраическая структура, автоморфная алгебраическая структура, универсальная алгебраическая структура, фундаментальные поля.

THE ALGEBRODYNAMICS: IN SEARCH OF THE ULTIMATE ALGEBRAIC “WORLD” STRUCTURE

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Principles of the so-called *algebrodynamical* approach to the construction of a unified field theory are presented, together with realization of the approach on the base of the linear algebra of *complex quaternions*. Then we discuss possible realizations of the algebrodynamics on a manifold equipped with the structure of a Lie group or its specific generalizations – algebraic structures (AS) defined by a *single operation* subject to a *single relation* containing three or four elements (analogous to the associativity requirement for a Lie group). Defined in such a way the so-called *invariant AS* turns out to be equivalent to a Lie group but allows thus for a non-canonical introduction of the latter which makes use of a single defining relation. The two more types of remarkable AS, the so-called *automorphic* and *universal* ones, are proposed for the role of the “World AS” and preliminary examined. Fundamental physical fields $F(x)$ are considered as *nontrivial mappings* of the elements of AS corresponding, in particular, to the multiplication of any element by itself, $F(x) = x \cdot x$.

Keywords: Biquaternions, Lie groups, invariant algebraic structure, automorphic algebraic structure, universal algebraic structure, fundamental fields).

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Introduction. The algebrodynamics

It is generally accepted that the most trustful approach to the construction of a unified field theory is the *geometrodynamics* (GD). Principles of GD have been formulated by Clifford, Einstein, Weyl and

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Wheeler. In the GD paradigm all fundamental physical fields have purely geometrical origin, that is either are constructed from characteristics like curvature and torsion or are themselves vector (tensor) fields on the space-time manifold – geodesic, covariantly constant ones, etc.

However, the choice of the space-time geometry, – its dimension, topology, differential structure, – is completely phenomenological since none general criteria of such a choice appealing to the internal properties of space-time manifold have been established. Besides, even on a fixed geometrical background determination of the structures responsible for the physical fields themselves and their equations remains indefinite. Owing, in particular, to these factors the GD approach does not yet get substantial development.

On the other hand, it is known that algebraic structures defined on a manifold (the so-called "manifolds with a multiplication" [1, 2], naturally give rise to a corresponding geometry of the manifold. For example, a "Kleinian" group of the isometries defining metrical structure on the manifold should be isomorphic to the *group of the automorphisms* of the primary algebraic structure.

Importantly, among various classes of algebraic structures there exist representatives *exceptional* in their internal properties: three remarkable linear algebras (complex numbers, quaternions and octaves), five exceptional Lie groups (F_4, G_2, E_6, E_7, E_8), etc. Finally, any algebraic structure naturally defines some *mappings* on the manifold so that corresponding functions can be regarded as physical fields subject to certain functional-differential equations. In view of the above stated considerations, such *algebrodynamical* (AD) approach seems to be more substantiated and promising than the GD one.

The most elaborated version of the AD approach is based on the exceptional *linear* algebra of quaternions, precisely, on its complexification – the algebra of *biquaternions* $\mathbb{B}$. In this framework, the complex vector space of $\mathbb{B}$ naturally maps into the interior of the light cone of the Minkowski space [3] realizing the isomorphism between the spinor Lorentz group $SL(2,\mathbb{C})$ and the group $SO(3,\mathbb{C})$, – the automorphism group of $\mathbb{B}$.

As for the mappings-fields $F(Z)$ – functions of the $\mathbb{B}$-variable Z, in our approach [4, 5] (see also [6, 7] and references therein), these functions have been defined by *the conditions of* $\mathbb{B}$*-differentiability*

$$dF = \Phi \cdot dZ \cdot \Psi, \tag{0.1}$$

($\cdot$) being multiplication in $\mathbb{B}$. Conditions (0.1) represent themselves a natural generalization of the *holomorphy* conditions for the functions of complex variable and lead, as opposed to the Cauchy-Riemann equations, to the *nonlinear* differential equation of *complex eikonal* (see, e.g., [4, 5, 6, 7] for detail). In this connection, *nonlinearity* responsible for the self-interaction of corresponding physical fields arises as a direct consequence of the *non-commutativity* property of the quaternion-type algebras in question.

Under the reduction $Z \mapsto X$ onto the Minkowski subspace defined by *Hermitian* matrices $X = X^+$, equations (0.1) become Lorentz invariant and, moreover, acquire natural spinor (twistor) and gauge (self-dual) structures. The first property allows to obtain general solution of (0.1) in the form of an implicit algebraic equation on the components of $\mathbb{B}$-field, while self-duality guarantees the fulfillment of the equations for gauge Maxwell and $SL(2,\mathbb{C})$ Yang-Mills fields on any solution to the primary system of equations corresponding to (0.1).

Finally, (elementary) *particles* can be identified with singular points of corresponding mappings-functions defined by the singular loci of the Maxwell and Yang-Mills field strengths. Their spacial distribution and temporary dynamics are fully controlled by the same primary conditions (0.1) of $\mathbb{B}$-differentiability.

Thus, one manages to construct a substantive algebraic field/particle theory making use only of the properties of the exceptional [1] linear algebra $\mathbb{B}$ and the conditions of differentiability of $\mathbb{B}$-functions treated as physical fields.

However, any *linear* algebra, apart of the principal multiplication of elements, carries two additional operations inherited from the structure of its basic vector space (addition of vectors and multiplication

[1]Precisely, of the direct sum of two exceptional algebras, complex numbers and Hamiltonian quaternions

of those by a number). Therefore, below we consider whether this linear structure could be replaced by a "nonlinear" one defined by a single operation of "multiplication" of the points of a manifold and inducing, in the turn, geometry of the latter.

1. Lie group as a simplest nonlinear algebraic structure

The most interesting and having a lot of applications algebraic structure [2] is, of course, the structure of a continuous group, *the Lie group*.

The structure of an abstract group is given by three well-known postulates: associativity, existence of the unit and inverse elements. It is completely determined by corresponding linear algebra, the Lie algebra, with a set of structure constants $C^{\rho}_{\mu\nu}$, skew symmetric in low indices and satisfying the known Yacobi identity. Classification of Lie groups obtained by Eli Cartan includes, besides some infinite series, 5 well-known exceptional groups.

Geometry of the Lie groups' manifolds is closely related to the existence of the so-called right- (left-) invariant vector fields $v^{\nu}_{\mu}(x)$, defined through the multiplication of an element x by an inverse to the infinitesimally close to it element y, with coordinates $y^{\mu} = x^{\mu} + dx^{\mu}$,

$$f^{\mu} := (x \cdot y^{-1})^{\mu} = e^{\mu} + v^{\mu}_{\nu}(x)dx^{\nu}, \tag{1.1}$$

where e^{μ} are the coordinates of the unit element of the group. Making use of the properties of associativity and invertibility, one easily obtains that

$$v^{\mu}_{\nu}(f)df^{\nu} = v^{\mu}_{\nu}(x)dx^{\nu} = invariant, \tag{1.2}$$

while from the integrability conditions to the latter "invariance relation" (1.2) *the Maurer-Cartan equations* do follow,

$$\partial_{\mu}v^{\rho}_{\nu} - \partial_{\nu}v^{\rho}_{\mu} = C^{\rho}_{\alpha\beta}v^{\alpha}_{\mu}v^{\beta}_{\nu}. \tag{1.3}$$

As it was demonstrated in [4], the structure of vector fields allows for definition of the strength tensor of an effective field of the Yang-Mills type which, however, turns to zero by virtue of the Maurer-Cartan equations (1.3). Therefore, in [4, ch. 4] we proposed to consider matter as a sort of *invariant deformation of a fundamental algebraic structure.*

Below we shall discuss some promising algebraic structures which in a sense generalize the structure of a Lie group and pretend for the role of the "World structure". In this connection, we note that numerous known generalizations of the group structure via the denial of the associativity property or the existence of the unit element are too indefinite and do not lead to some noticeable new results (see, e.g., [8]). Similar situation takes place under the denial of the existence of the inverse operation, the "division", that is, under the transition to the structure of the so-called *semi-group.* Threfore, we need to formulate some alternative approach to define the algebraic "World structure" *exceptional in its internal properties.*

2. "Invariant" algebraic structure and a novel introduction of an abstract Lie group

Let us consider in more detail the property of invariance (1.2) of vector fields on a Lie group. In fact, the latter is based on the following relation valid for any three elements of the group

$$(x \cdot z^{-1}) \cdot (y \cdot z^{-1})^{-1} = (x \cdot y^{-1}), \tag{2.1}$$

to prove which one should exploit all of the three properties of the group multiplication, that is, associativity and existence of the unit and inverse elements.

In this connection, it seems promising to introduce a new algebraic structure on a manifold **M** defined by a *single* invariance postulate reproducing (2.1). Consider, therefore, an algebraic structure on

[2] For the first turn, this is related to the description of symmetries of systems or processes by continuous *groups of transformations.*

a manifold $\mathbf{M}$ for which only one operation is defined. Below, for convenience, instead of the habitual "multiplication" $(\cdot)$ we shall treat this operation as "substraction" and denote it hereafter as $(-)$. Besides usual topological assumptions of continuity and smoothness of the operation, to define the properties of the sought-for structure we require for any three elements $x, y, z \in \mathbf{M}$ the following relation to be fulfilled:

$$(x - z) - (y - z) = x - y, \tag{2.2}$$

that is, the property of invariance of the principal operation of substraction w.r.t. a *shift* by an arbitrary element z. We shall call the structure defined by (2.2) the *invariant algebraic structure (IAS)* and examine whether it induces a (non-associative) generalization of the Lie group structure.

Notice firstly that from (2.2) it follows immediately

$$(x - z) - (x - z) = x - x, \quad \Rightarrow G(x - z) = G(x), \tag{2.3}$$

where a mapping $G : x \mapsto (x - x)$ is introduced. Since z is arbitrary element, from relation (2.3) it follows that $G(x) = \mathbf{0}$, where $\mathbf{0}$ is a *universal ("null") element* of the algebraic structure in question. Thus, existence of the null element (the direct analogue of the null (unit) element in the Lie group, see below) should not be postulated but follows from the principal relation (2.2) defining the IAS.

After this, one can automatically define, for any $x \in \mathbf{M}$, the element $\bar{x}$, *opposite* to x,

$$\bar{x} := \mathbf{0} - x, \tag{2.4}$$

and introduce a supplementary operation of "addition" $(+)$ for any pair of elements $x, y \in \mathbf{M}$,

$$x + y := x - \bar{y}. \tag{2.5}$$

The null element and introduced operations have the following properties for any elements $x, y, z, \ldots \in \mathbf{M}$. These properties are the consequences of the principal relation (2.2) and proved in the Appendix:

- (a) $\bar{\mathbf{0}} = \mathbf{0}$,
- (b) $x - \mathbf{0} = x$,
- (c) $\bar{\bar{x}} \equiv \mathbf{0} - \bar{x} = x$,
- (d) $x + \bar{x} = \mathbf{0}$,
- (e) $x + \mathbf{0} = \mathbf{0} + x = x$,
- (f) $\overline{x - y} = y - x$,
- (g) $(x - \bar{y}) - y = x$.

Using these properties, one can prove the associativity of the addition operation $(+)$,

$$(x + y) + z = x + (y + z). \tag{2.6}$$

Note that (2.6) can be equivalently represented in the form $(x - \bar{y}) - \bar{z} = x - (\overline{y - \bar{z}})$ or, using property (f),

$$(x - \bar{y}) - \bar{z} = x - (\bar{z} - y). \tag{2.7}$$

To prove (2.7), let us rewrite the principal relation (2.2) as

$$(x - \bar{y}) - \bar{z} = x - w, \tag{2.8}$$

where $\bar{z} := w - \bar{y}$. From the last definition one obtains $\bar{z} - y = (w - \bar{y}) - y$ or, using property (g), $\bar{z} - y = w$. Therefore, the principal relation (2.8) aquires the souught-for form (2.7), and the associativity property (2.6) is proved.

We come thus to the conclusion that the IAS is not essentially a presupposed generalization of the group. Nonetheless, we have obtained the following remarkable result. *The structure of "multiplication" for any Lie group can be uniquely introduced through a single operation of "substraction" which is completely defined by a single requirement of invariance (2.2).*

To conclude, it is quite evident that the operation $(+)$ completely reproducing the canonical multiplication in the Lie group structure allows for the inverse operation, so that the equations $a+x=b$ and $x+a=b$ have unique solutions $\forall a,b \in \mathbf{M}$. Specifically, from the first equation $x=b+\bar{a}$ while for the second the solution is $x=b+\bar{a}$. It is also easy to prove that the unique solution of the equation $a-x=b$ is given by $x=\bar{b}-\bar{a}\equiv\bar{b}+a$ while the solution of the equation $x-a=b$ is $x=b-\bar{a}\equiv b+a$.

Note finally that from the decomposition of $(x-y)$ in the vicinity of the null element which follows explicitly from (2.2),

$$(x-y)^\mu \sim x^\mu - y^\mu + b^\mu_{(\nu\rho)}(x^\nu - y^\nu)y^\rho + c^\mu_{[\nu\rho]}x^\nu y^\rho + \dots, \tag{2.9}$$

(where in the r.h.s. the sign $(-)$ has usual arithmetical sense) it follows that the IAS is determined by a set of structural constants $c^\mu_{[\nu\rho]}$ of some linear algebra subject to the Yacobi identity and isomorphic to a linear *Lie algebra*.

3. "Automorphic" algebraic structures and fundamental mappings - fields

As another possible candidate for the role of the "World algebraic structure" let us consider the defining relation of the following form:

$$(x-z)-(y-z)=(x-y)-z, \tag{3.1}$$

for any three elements of the sought-for struture $x,y,z\in\mathbf{M}$. The latter can be naturally called an *automorphic algebraic structure* (AAS), since the mapping $F: x\mapsto(x-z)$ is the automorphism of the AAS itself, that is,

$$F(x)-F(y)=F(x-y). \tag{3.2}$$

Remarkably, the AAS defining relation (3.1) is satisfied if one defines the principal operation of substraction $(-)$ through the "multiplication" $(\cdot)$ on a complementary structure of a Lie group. Specifically, one can set

$$x-y := y\cdot x^{-1}\cdot y. \tag{3.3}$$

Then the mapping $G: x\mapsto x-x$ becomes an identity, $G(x)=x$, one more relation takes place in addition

$$(y-x)-x=y \tag{3.4}$$

and, moreover, under some refined assumptions the structure of AAS comes into correspondence with the structure of a *symmetric space*. The latter can be algebraically defined through the postulate of "right- (left-) distributivity" like (3.1), inversibility (3.4) and idempotentivity $G(x):=x-x=x$ (see, e.g., [1, 9]).

In the turn, it is well known that the structure of symmetric spaces themselves is closely related to that of the Lie groups. On this base, the complete classification of symmetric spaces has been obtained by E. Cartan (see, e.g., [1, 2]).

Generally, however, the property of idempotentivity *does not* follow from the principal relation (3.1), whereas the alternative assumption on the mapping of *any* element $G(x)=x-x$ into a *universal (null) element* is immediately proved to be contradictory. Therefore, one can regard $G(x)$ as a nontrvial, *point dependent* mapping whose structure could define the AAS itself and corresponding geometry of the manifold as well. Physically, it would be natural to identify the form of this mapping with the structure of *fundamental fields* analogous to that of differentiable $\mathbb{B}$-functions in the framework of biquaternionic algebrodynamics (see section 1).

Thus, the AAS is richer in its internal properties than the group structure and only in a particular case reduces to the structure of a symmetric space in fact isomorphic to the Lie group structure. There exists a number of ways to define, on the AAS manifold, the structure of fundamental physical fields, the above proposed among them (see also [4, ch.3]). Corresponding differential equations for those fields as well as linear algebras defining the AAS, are the subject of further investigations.

4. "Universal" algebraic structure

IFinally, let us consider one more remarkable algebraic structure which can be defined by a single relation for any *four* elements $x, x', y, y' \in \mathbf{M}$,

$$(x' - y') - (x - y) = (x' - x) - (y' - y), \tag{4.1}$$

which corresponds, conditionally, to the following formulation by words: *"increment of differences equals to difference of increments"*. We shall call this structure *universal* algebraic structure (UAS).

Setting now $x' = x,\ y' = y$, one has for the function $G(x) = x - x$:

$$G(x - y) = G(x) - G(y). \tag{4.2}$$

This functional equation has two evident solutions, $G(x) = \mathbf{0}$ и $G(x) = x$, independently of the algebraic structure itself. In the first case, $G(x) = \mathbf{0}$, the UAS can reduce to the IAS and, thus, to the equivalent structure of a Lie group. In the second case, $G(x) = x$, the reduction to the structure of a symmetric space is possible. We, however, shall assume as above that the mapping $G : x \mapsto x - x$ is nontrivial and can be interpreted as a primary physical field.

We can speculate a bit about the properties of such fundamental structure. Specifically, the mapping G can possess immobile $G(x_0) = x_0$ or, more generally, cyclic $G(G(...G(x_0))..) = x_0$ points. One can also presuppose the existence of domains on the basic manifold whose points are mapped in the procedure into vicinity of a universal (null) element. Note that, generally, such element can be not unique.

Possible physical interpretation of the cycles and the set of null elements arising in the procedure devotes special discussion. In any case, simplicity of the definition of UAS, its uniqueness and richness of possibilities makes it, from our point of view, *the most suitable candidate to the role of the "World algebraic structure"*.

Conclusion

In the paper we have introduced a number of algebraic structures on a manifold very simple in the definition (through a single algebraic connection of the elements) and remarkable in their internal properties. The so-called invariant algebraic structure was proved to be isomorphic to the structure of a Lie group (which, therefore, can be introduced in an elegant, non-canonical way). The two other structures seem to be rather complicated and need further investigation as well as the introduction of physical fields and geometry of the manifold consistent with the properties of the primary algebraic structure. We think, nonetheless, that the novel types of structures proposed in the paper can stimulate the search of actually "World algebra" which could determine both the ultimate geometry of space-time and the dynamics of fundamental physical field(s).

5. Appendix

Let us prove the properties (a) – (g) of the IAS structure written out in section 3. Relation (a) follows directly from (2.2), that is, $\mathbf{0} \equiv \mathbf{0} - \mathbf{0} = (x - x) - (x - x) = x - x = \mathbf{0}$.

For relation (b) we take $z = \bar{y}$ and rewrite (2.2) in the form $(x - \bar{y}) - (\bar{y} - \bar{y}) = x - \bar{y}$, so that $(x - \bar{y}) - \mathbf{0} = x - \bar{y}$ or, taking $x = \mathbf{0}$, obtain $y - \mathbf{0} = y, \quad \forall y \in \mathbf{M}$.

As for relation (c), one has $\bar{\bar{x}} \equiv \mathbf{0} - (\mathbf{0} - x) = (x - x) - (\mathbf{0} - x)$ and, in view of (2.2) and (b), $\bar{\bar{x}} = x - \mathbf{0} = x$.

For (d) one obtains immediately $x + \bar{x} = x - \bar{\bar{x}} = x - x = \mathbf{0}$.

Now, (e) follows as $x + \mathbf{0} = x - \bar{\mathbf{0}} = x - \mathbf{0} = x$ and, similarly, $\mathbf{0} + x = \mathbf{0} - \bar{x} = \bar{\bar{x}} = x$.

To prove (f) we take $z = x$ in (2.2) and rewrite it then as $\mathbf{0} - (y - x) = x - y$, that is, $\overline{y - x} = x - y$.

Finally, for (g) one obtains using (2.2) and (b): $(x - \bar{y}) - y = (x - \bar{y}) - (\mathbf{0} - \bar{y}) = x - \mathbf{0} = x$, $\forall x, y \in \mathbf{M}$.

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