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ОБОБЩЕНИЕ ПРОЦЕДУРЫ СОГЛАСОВАНИЯ C^3

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Для решения проблем, связанных с присутствием разрывов вдоль совпадающих гиперповерхностей, в данной работе мы представляем обобщение процедуры совпадения C^3, рассмотренной в предыдущих работах. Они требуют, чтобы решение уравнений Эйнштейна также описывало совпадающую гиперповерхность.

Ключевые слова: Условия соответствия, соответствие C^3, собственные значения кривизны.

GENERALIZATION OF THE C^3 MATCHING PROCEDURE

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To handle the case in which discontinuities are present along the matching hypersurface, in this work, we present a generalization of the C^3 matching procedure discussed in previous works. It demands that a solution of Einstein's equations also describe the matching hypersurface.

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Introduction

General relativity is a theory of the gravitational interaction and, in particular, should describe the gravitational field of relativistic compact objects. In this case, the spacetime can be split into two different parts, namely, the interior region described by an exact solution of Einstein's equations with a physically reasonable energy-momentum tensor and the exterior region, which corresponds to an exact vacuum solution. This implies that the spacetime can be considered as split into two regions with certain hypersurface Σ at which the two regions should be matched.

This problem has been investigated for a long time [1]. In 1927, Darmois [2, 3] proposed that a physically meaningful matching can be obtained by demanding that the first and second fundamental forms (induced metric and extrinsic curvature, respectively) be continuous across Σ. Later on, in 1955, Lichnerowicz [4] proposed an alternative approach that turned out to be equivalent to the Darmois approach by choosing the underlying coordinates appropriately. If the fundamental forms are not

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continuous across the matching surface, Israel proposed in [5] to "cover" Σ with a shell, whose energy-momentum tensor takes care of the discontinuities. In the case of compact objects, Σ should be identified with the surface of the object, i.e., it is a time-like hypersurface. It then follows that at Σ, certain matching conditions should be imposed in order for the spacetime to be well defined.

Recently [6], we propose to use a C^3 criterion to find information about the location of the hypersurface Σ. It is defined in terms of the eigenvalues of the Riemann curvature tensor, which are invariant quantities. The idea is simple. Since the curvature tensor is a measure of the gravitational interaction, the curvature eigenvalues provide us with an invariant measure of the gravitational interaction. Since, for a compact object, one expects the spacetime to be asymptotically flat, the curvature eigenvalues should vanish at spatial infinite, and the behavior of the eigenvalues approaching the gravitational source could give some information about its borders.

In this work, we continue the investigation of the C^3 procedure. Based on Israel's formalism [5], we propose a general approach considering cases in which discontinuities are present along the matching surface Σ. It consists of demanding that a solution of Einstein equations also describe the 3-dimensional hypersurface Σ. In Section 1, we review in detail the main aspects of the C^3 matching procedure, whereas Section 2 is devoted to proposing a generalization of the C^3 matching procedure. Finally, in Section 3, we sum up our results.

1. C^3 matching procedure

The C^3 matching procedure is based on the analysis of the behavior of the Riemann curvature eigenvalues. This method was applied to study asymptotically flat spacetimes in [6]. Here, we employ the Cartan formalism of differential forms and local orthonormal tetrads to determine these eigenvalues. A local orthonormal tetrad is the simplest and most natural choice for an observer in order to perform local measurements of time, space, and gravity. So, let us choose the local ortho-normal tetrad ϑ^a, $a = 0, ..., 3$ such that

$$ds^2 = g_{\mu\nu} dx^\mu \otimes dx^\nu = \eta_{ab} \vartheta^a \otimes \vartheta^b \ , \tag{1.1}$$

with $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$, and $\vartheta^a = e^a{}_\mu dx^\mu$. The first and second Cartan equations

$$d\vartheta^a = -\omega^a{}_b \wedge \vartheta^b \ , \tag{1.2}$$

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = \frac{1}{2} R^a{}_{bcd} \vartheta^c \wedge \vartheta^d \tag{1.3}$$

allow us to compute the components of the Riemann curvature tensor R_{abcd} in the local orthonormal frame ϑ^a. Moreover, we define the Ricci tensor and the scalar curvature as $R_{ab} = R^c{}_{acb}$ and $R = R^a{}_a$, respectively. Furthermore, we introduce the bivector representation that consists in defining the curvature components R_{abcd} as the components of a 6×6 matrix $\mathbf{R}_{AB}$ according to the convention proposed in [7] (Chapter 14, Section 14.1, pp. 333-334), which establishes the following correspondence between tetrad ab and bivector indices A:

$$01 \to 1 \ , \quad 02 \to 2 \ , \quad 03 \to 3 \ , \quad 23 \to 4 \ , \quad 31 \to 5 \ , \quad 12 \to 6 \ . \tag{1.4}$$

Hence, the Riemann tensor can be represented as a symmetric matrix $\mathbf{R}_{AB}$ with 21 independent components. The algebraic Bianchi identity $R_{a[bcd]} = 0$, which in bivector representation reads

$$\mathbf{R}_{14} + \mathbf{R}_{25} + \mathbf{R}_{36} = 0 \tag{1.5}$$

reduces the number of independent components to 20. Furthermore, Einstein's equations, in geometric units such that $k = 8\pi G c^{-4}, G = c = 1$,

$$R_{ab} - \frac{1}{2} R \eta_{ab} = k \, T_{ab} \ , \tag{1.6}$$

can be written explicitly in terms of the curvature components $\mathbf{R}_{AB}$, resulting in a set of ten algebraic equations that relate the components of $\mathbf{R}_{AB}$ and T_{ab}. Consequently, only ten components $\mathbf{R}_{AB}$ are algebraic independent and can be arranged in the 6×6 curvature matrix in the following way

$$\mathbf{R}_{AB} = \begin{pmatrix} \mathbf{M}_1 & \mathbf{L} \\ \mathbf{L} & \mathbf{M}_2 \end{pmatrix}, \tag{1.7}$$

where

$$\mathbf{L} = \begin{pmatrix} \mathbf{R}_{14} & \mathbf{R}_{15} & \mathbf{R}_{16} \\ \mathbf{R}_{15} - kT_{03} & \mathbf{R}_{25} & \mathbf{R}_{26} \\ \mathbf{R}_{16} + kT_{02} & \mathbf{R}_{26} - kT_{01} & -\mathbf{R}_{14} - \mathbf{R}_{25} \end{pmatrix},$$

and $\mathbf{M}_1$ and $\mathbf{M}_2$ are 3×3 symmetric matrices

$$\mathbf{M}_1 = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ \mathbf{R}_{12} & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{R}_{13} & \mathbf{R}_{23} & -\mathbf{R}_{11} - \mathbf{R}_{22} + k\left(\frac{T}{2} + T_{00}\right) \end{pmatrix},$$

$$\mathbf{M}_2 = \begin{pmatrix} -\mathbf{R}_{11} + k\left(\frac{T}{2} + T_{00} - T_{11}\right) & -\mathbf{R}_{12} - kT_{12} & -\mathbf{R}_{13} - kT_{13} \\ -\mathbf{R}_{12} - kT_{12} & -\mathbf{R}_{22} + k\left(\frac{T}{2} + T_{00} - T_{22}\right) & -\mathbf{R}_{23} - kT_{23} \\ -\mathbf{R}_{13} - kT_{13} & -\mathbf{R}_{23} - kT_{23} & \mathbf{R}_{11} + \mathbf{R}_{22} - kT_{33} \end{pmatrix},$$

with $T = \eta^{ab} T_{ab}$. This is the most general form of a curvature tensor that satisfies Einstein's equations with an arbitrary energy-momentum tensor. The eigenvalues λ_n $(n = 1, \cdots, 6)$ of the matrix $\mathbf{R}_{AB}$ are known as the curvature eigenvalues.

The invariant character of the curvature eigenvalues allows us to apply them in many different physical situations and configurations. In particular, we here use the eigenvalues to match asymptotically flat solutions of the Einstein equation to its interior counterpart describing a material source of the gravitational field. In the C^3 matching approach, the matching surface Σ is determined by the matching radius, r_{match}, defined as

$$r_{match} \in [r_{rep}, \infty) \ , \quad r_{rep} = \max\{r_l\} \ , \tag{1.8}$$

where r_l $(l = 1, 2, ...)$, with $0 < r_l < \infty$, represents the set of solutions of the equation

$$\left.\frac{\partial \lambda_n^+}{\partial r}\right|_{r=r_l} = 0 \ , \tag{1.9}$$

with λ_n^+ being the curvature eigenvalues of the manifold $(\mathcal{M}^+, \mathbf{g}^+)$, which is assumed to be asymptotically flat, i.e., there exists a spatial coordinate r such that

$$\lim_{r \to \infty} \mathbf{g}^+ = \eta \tag{1.10}$$

where η represents the Minkowski metric.

Theorem 1.1. *Let $(\mathcal{M}^-, \mathbf{g}^-)$ and $(\mathcal{M}^+, \mathbf{g}^+)$ be an arbitrary and an asymptotically flat spacetime, which satisfy Einstein equations, and let λ_n^- and λ_n^+ be the curvature eigenvalues of $(\mathcal{M}^-, \mathbf{g}^-)$ and $(\mathcal{M}^+, \mathbf{g}^+)$, respectively. Then, we say that $\mathcal{M}^-$ and $\mathcal{M}^+$ can be matched at the surface Σ, determined by the matching radius r_{match} as defined in Eq.(1.8), if the necessary and sufficient condition*

$$[\lambda_n] \equiv \lambda_n^- - \lambda_n^+ = 0, \quad n = 1, \cdots, 6 \tag{1.11}$$

is satisfied.

From a pragmatical point of view, the interior region of compact objects corresponds to the spacetime $(\mathcal{M}^-, \mathbf{g}^-)$ whereas the exterior region is described by $(\mathcal{M}^+, \mathbf{g}^+)$. Then, r_l represent the extrema of the exterior eigenvalues λ_n^+ and the repulsion radius r_{rep} corresponds to the extremum r_l with the maximum value. In other words, r_{rep} is the value of r, where the first extremum of λ_n^+ is encountered when approaching the origin of coordinates $r = 0$ coming from infinity. The spacetimes $(\mathcal{M}^+, \mathbf{g}^+)$ and $(\mathcal{M}^-, \mathbf{g}^-)$ can be matched at the matching radius r_{match}, which can be chosen at any value of r located between the repulsion radius r_{rep} and infinity.

2. C^3 matching across spherically symmetric thin shells

In general relativity, the interior of spacetimes corresponding to a spherically symmetric perfect fluid can be described by the energy-momentum tensor

$$\mathbf{T}^{\alpha\beta} = (\rho + p)\mathbf{V}^{\alpha}\mathbf{V}^{\beta} + p\mathbf{g}^{\alpha\beta} \tag{2.1}$$

where ρ and p are the energy density and the pressure of the fluid, respectively, and $\mathbf{V}$ is the velocity of the fluid, which we choose as the comoving velocity $\mathbf{V}_{\alpha} = (-1, 0, 0, 0)$.

According to Birkhoff's theorem, the exterior spacetime must be described by the Schwarzschild metric

$$\mathbf{g}^{+} = -\left(1 - \frac{2m}{r}\right) \mathrm{d}\, t \otimes \mathrm{d}\, t + \left(1 - \frac{2m}{r}\right)^{-1} \mathrm{d}\, r \otimes \mathrm{d}\, r + r^2(\mathrm{d}\, \theta \otimes \mathrm{d}\, \theta + \sin^2\theta\, \mathrm{d}\, \phi \otimes \mathrm{d}\, \phi). \tag{2.2}$$

Furthermore, we choose the matching hypersurface as a sphere of constant radius. For the C^3 approach we only need to calculate the curvature eigenvalues. We choose the orthonormal tetrad ϑ^a as

$$\vartheta^0 = \left(1 - \frac{2m}{r}\right)^{1/2} \mathrm{d}\, t\ ,\ \vartheta^1 = \left(1 - \frac{2m}{r}\right)^{-1/2} \mathrm{d}\, r\ ,\ \vartheta^2 = r\, \mathrm{d}\, \theta\ ,\ \vartheta^3 = r \sin\theta\ \mathrm{d}\, \phi\ . \tag{2.3}$$

A straightforward computation shows that the curvature matrix $\mathbf{R}_{AB}$ is diagonal and the eigeinvalues are

$$\lambda_2^{+} = \lambda_3^{+} = -\lambda_5^{+} = -\lambda_6^{+} = m/r^3, \quad \lambda_1^{+} = -\lambda_4^{+} = -2m/r^3. \tag{2.4}$$

To perform the C^3 procedure, we first find the extrema of the exterior eigenvalues. As we can see, none of the Schwarzchild eigenvalues has an extremum. This means that there is no repulsion radius r_{rep}, which indicates in the approach the smallest sphere at which the matching can be carried out. Consequently, there is no repulsion region in the Schwarzschild spacetime that should be covered by an interior solution, which is the conceptual background of the C^3 approach. Then, the matching radius can be located anywhere outside the central singularity, i.e., $r_{match} \in (0, \infty)$.

In previous references (See [6, 8]), it was shown that in the case of spherically symmetric perfect fluids, the vanishing of the energy-momentum tensor on the matching surface is a necessary condition to perform the matching procedure; this suggests that the jump of the curvature eigenvalues across the matching surface vanishes. Recently, we have glimpsed particular solutions corresponding to perfect fluids in which this does not occur. In this work, we construct a formalism that allows the C^3 matching in the case of discontinuities across the matching surface, i.e., $\lambda_n^{+} \neq \lambda_n^{-}$ on Σ for at least one value of n.

We will use Israel's formalism [5] as a conceptual guide that allows the existence of discontinuities of the first and second fundamental forms by introducing an effective energy-momentum tensor on the matching surface Σ so that it can be intepreted as a infinitesimal matter shell that join the interior and exterior spacetimes. To this end, let us consider the jump of the eigenvalues across Σ as

$$[\lambda_n] = \lambda_n^{-} - \lambda_n^{+}\ . \tag{2.5}$$

In the case of a matching between an interior perfect fluid solution and the exterior Schwarzschild vacuum solution, we have shown that the C^3 procedure implies that ρ and p should be zero on Σ. When these conditions are not satisfied, let us define the surface density σ and pressure π as

$$\sigma = \rho|_{\Sigma}\ , \qquad P = p|_{\Sigma}\ . \tag{2.6}$$

Then, since in the case of discontinuities we have that $[\lambda_n] \neq 0$, it follows that $\sigma \neq 0$ and $P \neq 0$, in general. This is equivalent to saying that the explicit values of $[\lambda_n]$ should contain information about the physical quantities σ and P. For this reason, we assume that $[\lambda_n]$ is arbitrary in value but finite.

The question is now whether σ and P can be used to construct a realistic matter shell on Σ. To this end, consider the jump of the Einstein tensor on Σ, i. e.,

$$[G_{ij}] = G^-_{ij} - G^+_{ij} \ , \quad G^\pm_{ij} = \frac{\partial x^\mu_\pm}{\partial \xi^i}\frac{\partial x^\nu_\pm}{\partial \xi^j} G^\pm_{\mu\nu} \ , \tag{2.7}$$

where ξ^i are the coordinates of the surface Σ and $x^\mu_\pm$ are the coordinates of the interior and exterior spacetimes, respectively. Then, $G^\pm_{ij}$ is the Einstein tensor induced on Σ. Furthermore, we introduce an energy-momentum tensor S_{ij} on Σ as

$$[G_{ij}] = kS_{ij} \ . \tag{2.8}$$

Certainly, it is always possible to introduce algebraically an energy-momentum tensor in this way. However, the essential point is whether S_{ij} is physically meaningful. To guarantee the fulfillment of this condition, we demand that S_{ij} be induced by the energy-momentum tensors of the interior and exterior spacetimes and be in agreement with their physical significance. Then, in the case of the perfect fluid we are considering here, we demand that

$$S_{ij} = [T_{ij}] = T^-_{ij} - T^+_{ij} = (\sigma + P)u_i u_j + P\gamma_{ij} \ , \tag{2.9}$$

where $T^\pm_{ij}$ are the energy-momentum tensors and $\gamma_{ij} = \gamma^\pm_{ij}$ is the metric tensor induced on Σ, respectively.

In summary, in the case of discontinuities, we will say that an interior spacetime can be matched with an exterior one along a boundary shell located on Σ, if there exist a density σ and a pressure P, satisfying the induced Einstein equations (2.8) and (2.9) and the boundary condition (2.6).

In the case of spherical symmetry the coordinates on both sides of the boundary can be chosen as $x^\mu_\pm = (t, r, \theta, \phi)$ and on the matching surface as $\xi^i = (t, \theta, \phi)$. Then, all the components of the quantities $\partial x^\mu / \partial \xi^i$ are constant and the induced tensors can be calculated in a straightforward way. We obtain for the jump of the eigenvalues along the matching surface $r = r_{\rm match}$ the following expressions

$$[\lambda_2] = [\lambda_3] = [\lambda_4] = 0 \ , \quad [\lambda_1] = [\lambda_5] = [\lambda_6] = 4\pi\sigma \ , \tag{2.10}$$

which agrees with the result that on the matching surface the pressure vanishes. Furthermore, the jump of components of the induced Einstein tensor can be expressed as

$$[G_{ij}] = 2k[\lambda_1]u_i u_j = k\sigma u_i u_j \ , \quad u^i = (-1, 0, 0) \ . \tag{2.11}$$

It is then easy to see that on the matching surface $r = r_{\rm match}$, the induced Einstein equations for dust are satisfied, proving that, in fact, a realistic dust shell can be introduced that allows us to match, in the framework of the C^3 matching procedure, perfect fluids with the exterior Schwarzschild spacetime.

3. Conclusions

In this work, we have analyzed the problem of matching spherically symmetric solutions of Einstein equations. n particular, we limit ourselves to the case of interior solutions corresponding to perfect static fluids. We know that in specific perfect fluid solutions, the energy density shows a discontinuity across the matching surface; to handle the case in which discontinuities are present along the matching surface Σ, we propose in this work a generalization of the C^3 matching procedure. It consists of demanding that a solution of Einstein equations also describe the 3-dimensional hypersurface Σ. In fact, we consider the induced Einstein tensor on Σ and show that it can be represented as a realistic energy-momentum tensor that describes the matter inside a boundary shell located on Σ. In the cases considered in this work, it turned out that the boundary corresponds to a dust shell. For more general interior solutions, we expect to obtain shells with more intricate internal structures.

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