УДК 537.8

КВАЗИКЛАССИЧЕСКОЕ ОПИСАНИЕ ЭЛЕКТРОМАГНИТНОГО ИЗЛУЧЕНИЯ УСКОРЕННЫХ ЗАРЯДОВ*

Адорно Т. К.[a,b,1], Бреев А. И.[b,2], Гитман Д. М.[c,d,b,3]

[a] Факультет физики, Сианьский университет Цзяотун-Ливерпуль, 111 Ren'ai Road, Сучжоу, район науки и образования озера Душу, промышленный парк Сучжоу, Сучжоу, 215123, Китай.
[b] Томский государственный университет, пр. Ленина 36, Томск, 634050, Россия.
[c] Физический институт им. П. Н. Лебедева, Ленинский проспект 53, 119991, Москва, Россия.
[d] Институт физики, Университет Сан-Паулу, улица Матан, 1371, Сан-Паулу 05508-090, Бразилия.

Мы рассматриваем наши недавние результаты по электромагнитному излучению, создаваемому распределениями зарядов, в рамках квазиклассического подхода. В этом подходе токи, генерирующие излучение, рассматриваются классически, при этом точно сохраняется квантовая природа излучения. Величины, имеющие отношение к проблеме излучения, вычисляются с помощью вероятностей перехода, квантовые состояния которых электромагнитного поля имеют четко определенное число фотонов и являются решениями соответствующего уравнения Шредингера. Мы получим полную энергию и скорости, излучаемых точечных зарядов, ускоренными электромагнитными полями, и сравним полученные результаты с результатами, полученными в классической электродинамике. .

Ключевые слова: Классическая электродинамика, электромагнитное излучение, квантовая механика, уравнение Шредингера.

SEMICLASSICAL DESCRIPTION OF THE ELECTROMAGNETIC RADIATION BY ACCELERATED CHARGED DISTRIBUTIONS

Adorno T. C.[a,b,1], Breev A. I.[b,2], Gitman D. M.[c,d,b,3]

[a] Department of Physics, Xi'an Jiaotong-Liverpool University, 111 Ren'ai Road, Suzhou Dushu Lake Science and Education Innovation District, Suzhou Industrial Park, Suzhou, 215123, P. R. China.
[b] Department of Physics, Tomsk State University, Lenin Ave. 36, Tomsk 634050, Russia.
[c] P. N. Lebedev Physical Institute 53 Leninskiy prospekt, 119991 Moscow, Russia.
[d] Instituto de Física, Universidade de São Paulo, Rua do Matão, 1371, São Paulo 05508-090, Brazil.

We review our recent results on the electromagnetic radiation produced by charge distributions in the framework of a semiclassical approach. In this approach, currents, generating the radiation are considered classically, while the quantum nature of the radiation is kept exactly. Pertinent quantities to the radiation problem are calculated with the aid of transition probabilities, whose quantum states of the electromagnetic field have well-defined number of photons and are solutions of the corresponding Schrödinger's equation. We summarize the calculation of the total energy and and rates radiated by point charges accelerated by electromagnetic fields and compare our results with those obtained in classical electrodynamics.

Keywords: Classical electrodynamics, electromagnetic radiation, quantum mechanics, Schrödinger equation.

PACS: 03.50.De, 03.70.+k, 03.65.Sq, 03.50.-z

DOI: 10.17238/issn2226-8812.2023.3-4.12-20

*The work of T. C. Adorno was supported by the XJTLU Research Development Funding, award no. RDF-21-02-056 and the work of A. I. Breev and D. M. Gitman was supported by Russian Science Foundation, grant no. 19-12-00042. D. M. Gitman thanks CNPq for permanent support.

[1]E-mail: Tiago.Adorno@xjtlu.edu.cn
[2]E-mail: breev@mail.tsu.ru
[3]E-mail: gitman@if.usp.br

Introduction

It is known that a change in the state of motion of charged particles is usually accompanied by electromagnetic radiation. The most prominent examples are the radiation of accelerated charges performing rectilinear or circular motions. While the accurate description of the process should be carried out in the framework of QED (as described in Refs. [1, 2, 3], for instance) there are numerous cases where the direct application of the QED formalism leads to technical difficulties. In such cases, one usually resorts to the classical approximation, which is as follows: the motion of charged particles is considered within the framework of classical (non-relativistic or relativistic) mechanics, then using Maxwell's equations, the electromagnetic field generated by this motion is restored (for example, in the form of Liénard-Wiechert potentials), and finally, the observed electromagnetic radiation generated by the motion of the charges is calculated as the energy flux determined by the Poynting-Heaviside theorem, see, e.g., Refs. [4, 5, 6, 7, 8]. However, it must be noted that such a way of calculating electromagnetic radiation hinges on certain assumptions; in our opinion, the most adequate discussion on this matter is in Stratton's book [9].

In this work we summarize our recent results [10, 11] on an alternative approach to calculate electromagnetic radiation by charged distributions, which does not suffer from the technical difficulties associated with the application of QED nor the assumptions underlying the classical theory. We call such a formulation the semiclassical approach. In this approach, currents generating the radiation are considered classically, whereas the quantum nature of the radiation is taken into account exactly. This possibility is based on the exact construction of quantum states of the electromagnetic field interacting with the mentioned classical currents and subsequent consistent application of QED methods for calculating the radiation. Universal formulas describing multiphoton radiation were derived. The approach does not require knowledge of the exact solutions of relativistic wave equations with external fields; hence technical difficulties associated with using the Furry picture do not arise. Moreover, the semiclassical approach can be applied to any trajectory performed by the particle, even including cases with backreaction, as these can be accounted for by solving the Lorentz equations with radiation-reaction terms. We note that in the framework of the semiclassical approach, one can directly calculate the radiation emitted from any trajectory of a charged particle, whereas, in QED, the technics of calculating photon transition amplitudes (say in the Furry picture) is adopted only for charge motions caused by external electromagnetic fields. However, a univocal correspondence between every charge trajectory and a corresponding electromagnetic field does not exist. The efficacy of the semiclassical approach was demonstrated in calculating synchrotron [10] and undulator [12] radiations. In this work we consider the Minkowski spacetime, $\eta_{\mu\nu} = \mathrm{diag}\,(+1,-1,-1,-1)$, parameterized by coordinates $x^\mu = \left(x^0 = ct, \mathbf{r}\right)$. Boldface letters denote three-dimensional vectors, e.g. $\mathbf{r} = \left(x^i,\ i = 1,2,3\right)$, and three-dimensional differentials denote volume integration measures, e.g., $d\mathbf{r} = dx^1 dx^2 dx^3$. Gaussian units are used.

1. Semiclassical description of electromagnetic radiation induced by classical currents

The state vector $|\Psi(t)\rangle$ of the quantized electromagnetic field interacting with a classical current satisfies the Schrödinger equation and an initial condition $|\Psi\rangle_{\mathrm{in}}$ at the initial time instant t_{in},

$$i\hbar\partial_t |\Psi(t)\rangle = \hat{H}(t)|\Psi(t)\rangle\,, \quad |\Psi(t_{\mathrm{in}})\rangle = |\Psi\rangle_{\mathrm{in}}\,. \tag{1.1}$$

Here, $\hat{H}(t)$ is the Hamiltonian of the quantized electromagnetic field $\hat{A}^\mu(x) = \Big(A^0(x), \hat{\mathbf{A}}(\mathbf{r})\Big)$ interacting with a classical four-current $j^\mu(x) = \left(j^0(x), \mathbf{j}(x)\right)$. The potential $\hat{\mathbf{A}}(\mathbf{r})$ splits into a transverse $\hat{\mathbf{A}}_\perp(\mathbf{r})$ and longitudinal parts $\hat{\mathbf{A}}_\parallel(\mathbf{r})$:

$$\begin{aligned}
&\hat{\mathbf{A}}(\mathbf{r}) = \hat{\mathbf{A}}_\perp(\mathbf{r}) + \hat{\mathbf{A}}_\parallel(\mathbf{r})\ , \\
&\hat{\mathbf{A}}_\perp(\mathbf{r}) = \delta_\perp \hat{\mathbf{A}}(\mathbf{r})\,, \quad \hat{\mathbf{A}}_\parallel(\mathbf{r}) = (1-\delta_\perp)\hat{\mathbf{A}}(\mathbf{r})\ , \\
&\mathrm{div}\hat{\mathbf{A}}_\perp(\mathbf{r}) = 0, \quad \mathrm{curl}\hat{\mathbf{A}}_\parallel(\mathbf{r}) = 0\ ,
\end{aligned}$$

in which $\delta_{\perp}^{sp} = \delta^{sp} - \Delta^{-1}\partial^{s}\partial^{p}$ denotes the transverse projection operator, $(\delta_{\perp}\hat{A}(\mathbf{r}))^{s} = \delta_{\perp}^{sp}A^{p}(\mathbf{r})$ (see Refs. [3, 13]). Sticking to the Coulomb gauge – in which the longitudinal degree of freedom of $\hat{\mathbf{A}}\left(\mathbf{r}\right)$ is absent and only transverse photons are present – $\hat{\mathbf{A}}\left(\mathbf{r}\right) = \hat{\mathbf{A}}_{\perp}\left(\mathbf{r}\right)$, and the scalar potential $A^{0}\left(x\right)$ is a non operatorial solution of the Poisson equation,

$$\Delta A^{0}\left(x\right) = -\frac{4\pi}{c}j^{0}\left(x\right) \rightarrow A^{0}\left(x\right) = \frac{1}{c}\int\frac{j^{0}\left(t,\mathbf{r}'\right)}{\left|\mathbf{r}-\mathbf{r}'\right|}d\mathbf{r}' \,, \tag{1.2}$$

while the operator of the vector potential $\hat{\mathbf{A}}\left(\mathbf{r}\right)$ reads:

$$\hat{\mathbf{A}}\left(\mathbf{r}\right) = \sqrt{4\pi\hbar c}\sum_{\lambda=1}^{2}\int\left[\hat{c}_{\mathbf{k}\lambda}\mathbf{f}_{\mathbf{k}\lambda}\left(\mathbf{r}\right) + \hat{c}_{\mathbf{k}\lambda}^{\dagger}\mathbf{f}_{\mathbf{k}\lambda}^{*}\left(\mathbf{r}\right)\right]d\mathbf{k}\,,$$
$$\mathbf{f}_{\mathbf{k}\lambda}\left(\mathbf{r}\right) = \frac{e^{i\mathbf{kr}}\boldsymbol{\epsilon}_{\mathbf{k}\lambda}}{\sqrt{2k_{0}\left(2\pi\right)^{3}}}\,,\quad k^{\mu} = \left(k_{0} = \frac{\omega}{c},\mathbf{k}\right),\quad k_{0} = \left|\mathbf{k}\right|\,. \tag{1.3}$$

Here, $\hat{c}_{\mathbf{k}\lambda}^{\dagger}$ and $\hat{c}_{\mathbf{k}\lambda}$ are creation and annihilation operators of free photons with wave vector $\mathbf{k}$ and polarizations λ,

$$\left[\hat{c}_{\mathbf{k}\lambda},\hat{c}_{\mathbf{k}'\lambda'}^{\dagger}\right]_{-} = \delta_{\lambda\lambda'}\delta\left(\mathbf{k}-\mathbf{k}'\right),\quad \left[\hat{c}_{\mathbf{k}\lambda},\hat{c}_{\mathbf{k}'\lambda'}\right]_{-} = \left[\hat{c}_{\mathbf{k}\lambda}^{\dagger},\hat{c}_{\mathbf{k}'\lambda'}^{\dagger}\right]_{-} = 0\,, \tag{1.4}$$

while $\boldsymbol{\epsilon}_{\mathbf{k}\lambda}$ are complex polarization vectors that obey the orthogonality and completeness relations:

$$\boldsymbol{\epsilon}_{\mathbf{k}\lambda}\boldsymbol{\epsilon}_{\mathbf{k}\lambda'}^{*} = \delta_{\lambda\lambda'},\quad \boldsymbol{\epsilon}_{\mathbf{k}\lambda}\mathbf{k} = 0,\quad \sum_{\lambda=1}^{2}\epsilon_{\mathbf{k}\lambda}^{i}\epsilon_{\mathbf{k}\lambda}^{j*} = \delta^{ij} - n^{i}n^{j},\quad n^{i} = \frac{k^{i}}{\left|\mathbf{k}\right|}\,. \tag{1.5}$$

The Hamiltonian $\hat{H}\left(t\right)$ reads:

$$\hat{H}\left(t\right) = \hat{H}_{\gamma} + \frac{1}{c}\int\left[\frac{1}{2}j_{0}\left(x\right)A^{0}\left(x\right) - \mathbf{j}\left(x\right)\hat{\mathbf{A}}\left(\mathbf{r}\right)\right]d\mathbf{r}\,,$$
$$\hat{H}_{\gamma} = \hbar c\sum_{\lambda=1}^{2}\int k_{0}\hat{c}_{\mathbf{k}\lambda}^{\dagger}\hat{c}_{\mathbf{k}\lambda}d\mathbf{k}\,. \tag{1.6}$$

One can demonstrate that a solution of equation (1.1) can be written as

$$\left|\Psi\left(t\right)\right\rangle = U\left(t,t_{\mathrm{in}}\right)\left|\Psi\right\rangle_{\mathrm{in}}\,, \tag{1.7}$$

where the evolution operator $U\left(t,t_{\mathrm{in}}\right)$ has the form [10, 11],

$$U\left(t,t_{\mathrm{in}}\right) = e^{i\phi\left(t,t_{\mathrm{in}}\right)}U_{\gamma}\left(t,t_{\mathrm{in}}\right)\mathcal{D}\left(y\right)\,,$$
$$\mathcal{D}\left(y\right) = \exp\sum_{\lambda=1}^{2}\int\left[y_{\mathbf{k}\lambda}\left(t,t_{\mathrm{in}}\right)\hat{c}_{\mathbf{k}\lambda}^{\dagger} - y_{\mathbf{k}\lambda}^{*}\left(t,t_{\mathrm{in}}\right)\hat{c}_{\mathbf{k}\lambda}\right]d\mathbf{k}\,,$$
$$y_{\mathbf{k}\lambda}\left(t,t_{\mathrm{in}}\right) = i\sqrt{\frac{4\pi}{\hbar c}}\int_{t_{\mathrm{in}}}^{t}dt'\int\mathbf{j}\left(x'\right)\mathbf{f}_{\mathbf{k}\lambda}^{*}\left(x',t_{\mathrm{in}}\right)d\mathbf{r}'\,. \tag{1.8}$$

and $\phi\left(t,t_{\mathrm{in}}\right)$ is a phase.

With this solution we can calculate transition amplitudes and probabilities between states with a definite number of photons. For example, the transition probability that the vacuum state $\left|0\right\rangle = \left|\Psi\right\rangle_{\mathrm{in}}$ evolves to a state with N photons

$$\left|\{N\}\right\rangle = \frac{1}{\sqrt{N!}}\prod_{a=1}^{N}\hat{c}_{\mathbf{k}_{a}\lambda_{a}}^{\dagger}\left|0\right\rangle\,, \tag{1.9}$$

after a time interval $\Delta t = t - t_{\rm in}$ reads [11]:

$$
\begin{aligned}
P\left(\{N\};t,t_{\rm in}\right) &= \left|\langle\{N\}\left|\Psi\left(t\right)\right\rangle\right|^2 \\
&= \frac{1}{N!}\prod_{a=1}^{N}\left|y_{\mathbf{k}_a\lambda_a}\left(t,t_{\rm in}\right)\right|^2 \exp\left[-\sum_{\lambda=1}^{2}\int\left|y_{\mathbf{k}\lambda}\left(t,t_{\rm in}\right)\right|^2 d\mathbf{k}\right]. \qquad (1.10)
\end{aligned}
$$

From the above probability and the energy of N photons

$$
W\left(\{N\}\right) = \hbar c\sum_{a=1}^{N} k_{0,a}\,, \quad k_{0,a} = \left|\mathbf{k}_a\right|\,, \qquad (1.11)
$$

the observe that the total electromagnetic energy of N photons radiated by the current, $W\left(N;t,t_{\rm in}\right)$, is the sum of energies (1.11) weighted by the probability $P\left(\{N\};t,t_{\rm in}\right)$:

$$
\begin{aligned}
W\left(N;t,t_{\rm in}\right) &= \prod_{a=1}^{N}\sum_{\lambda_a=1}^{2}\int d\mathbf{k}_a W\left(\{N\}\right)P\left(\{N\};t,t_{\rm in}\right) \\
&= \frac{W\left(1;t,t_{\rm in}\right)}{\left(N-1\right)!}\left[\sum_{\lambda=1}^{2}\int\left|y_{\mathbf{k}\lambda}\left(t,t_{\rm in}\right)\right|^2 d\mathbf{k}\right]^{N-1}. \qquad (1.12)
\end{aligned}
$$

The prefactor $W\left(1;t,t_{\rm in}\right)$ denotes the electromagnetic energy of one photon radiated by the external current

$$
W\left(1;t,t_{\rm in}\right) = \hbar c P\left(0;t,t_{\rm in}\right)\sum_{\lambda=1}^{2}\int k_0\left|y_{\mathbf{k}\lambda}\left(t,t_{\rm in}\right)\right|^2 d\mathbf{k}\,. \qquad (1.13)
$$

Finally, summing Eq. (1.12) over all the photons, we obtain the *total* electromagnetic energy radiated by the external current:

$$
W\left(t,t_{\rm in}\right) = \sum_{N=1}^{\infty} W\left(N;t,t_{\rm in}\right) = \hbar c\sum_{\lambda=1}^{2}\int k_0\left|y_{\mathbf{k}\lambda}\left(t,t_{\rm in}\right)\right|^2 d\mathbf{k}\,. \qquad (1.14)
$$

With these results, we may define the rate at which energy is emitted from the source by differentiating Eq. (1.14) with respect to time:

$$
w\left(t,t_{\rm in}\right) = \frac{\partial W\left(t,t_{\rm in}\right)}{\partial t} = 2\hbar c\,\mathrm{Re}\sum_{\lambda=1}^{2}\int k_0\left[y_{\mathbf{k}\lambda}\left(t,t_{\rm in}\right)\frac{\partial}{\partial t}y^*_{\mathbf{k}\lambda}\left(t,t_{\rm in}\right)\right]d\mathbf{k}\,. \qquad (1.15)
$$

Equations (1.14) and (1.15) can be alternatively expressed in terms of the current density distribution $\mathbf{j}\left(\mathbf{r}\right)$. To this end, we substitute the function $y_{\mathbf{k}\lambda}\left(t,t_{\rm in}\right)$ given by Eq. (1.8) into Eqs. (1.14), (1.15) and simplify the summations over λ with the aid the identities (1.5) to finally obtain:

$$
\begin{aligned}
W\left(t,t_{\rm in}\right) &= 4\pi^2\int\left|\mathbf{n}\times\left[\mathbf{n}\times\tilde{\mathbf{j}}\left(k;t,t_{\rm in}\right)\right]\right|^2 d\mathbf{k}\,, \\
w\left(t,t_{\rm in}\right) &= 2\left(2\pi\right)^{3/2}\mathrm{Re}\int e^{-ik_0ct}\left\{\tilde{\mathbf{j}}^*\left(\mathbf{k};t\right)\tilde{\mathbf{j}}\left(k;t,t_{\rm in}\right)\right. \\
&- \left.\left[\mathbf{n}\tilde{\mathbf{j}}^*\left(\mathbf{k};t\right)\right]\left[\mathbf{n}\tilde{\mathbf{j}}\left(k;t,t_{\rm in}\right)\right]\right\}d\mathbf{k}\,. \qquad (1.16)
\end{aligned}
$$

where

$$
\tilde{\mathbf{j}}\left(k;t,t_{\rm in}\right) = \frac{1}{\sqrt{2\pi}}\int_{t_{\rm in}}^{t} e^{ik_0ct'}\tilde{\mathbf{j}}\left(\mathbf{k};t'\right)dt'\,, \quad \tilde{\mathbf{j}}\left(\mathbf{k};t'\right) = \frac{1}{\left(2\pi\right)^{3/2}}\int e^{-i\mathbf{k}\mathbf{r}'}\mathbf{j}\left(x'\right)d\mathbf{r}'\,. \qquad (1.17)
$$

The total energy, in particular, coincides with the classical result [6, 7] in the limits $t_{\rm in}\to-\infty$, $t\to+\infty$, namely,

$$
W\left(+\infty,-\infty\right) = W_{\rm cl} = 4\pi^2\int\left|\mathbf{n}\times\left[\mathbf{n}\times\tilde{\mathbf{j}}\left(k\right)\right]\right|^2 d\mathbf{k}\,, \qquad (1.18)
$$

$$
\tilde{\mathbf{j}}\left(k\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ik_0ct'}\tilde{\mathbf{j}}\left(\mathbf{k};t'\right)dt'\,. \qquad (1.19)
$$

Here $\tilde{\mathbf{j}}\left(k\right)$ denotes the ordinary four-dimensional Fourier transform of the current density $\mathbf{j}\left(x\right)$.

2. Electromagnetic energies and rates radiated by accelerated charges

2.1. Point charge in a constant and homogeneous magnetic field

Consider a point charge (with algebraic charge q and mass m) moving with velocity $\mathbf{v}(t)$ along a circular trajectory with radius $\mathbf{R}$ in a constant and homogeneous magnetic field $\mathbf{B} = (0, 0, B)$. The corresponding current density has the form

$$j^0(x) = q\delta(\mathbf{r} - \mathbf{r}(t)) \ , \quad \mathbf{j}(x) = qc\boldsymbol{\beta}(t)\,\delta(\mathbf{r} - \mathbf{r}(t)) \ , \quad \boldsymbol{\beta}(t) = \frac{\mathbf{v}(t)}{c} \ , \tag{2.1}$$

where the position $\mathbf{r}(t)$ and the velocity read [8]

$$\mathbf{r}(t) = (R\cos\Omega t, R\sin\Omega t, 0) \ , \quad \mathbf{v}(t) = \Omega R(-\sin\Omega t, \cos\Omega t, 0) \ , \quad \Omega = \frac{eB}{mc} \, . \tag{2.2}$$

Plugging this current into Eq. (1.8) we find,

$$y_{\mathbf{k}\lambda}(t, t_{\text{in}}) = -\frac{iqc}{2\pi}\frac{e^{-ik_0ct_{\text{in}}}}{\sqrt{\hbar ck_0}}\int_{t_{\text{in}}}^{t} e^{i\Phi(t')}\boldsymbol{\beta}(t')\,\boldsymbol{\epsilon}^*_{\mathbf{k}\lambda}dt', \quad \Phi(t') = k_0ct' - \mathbf{kr}(t') \ . \tag{2.3}$$

Setting $t_{\text{in}} = 0$ and choosing a reference frame whose origin coincides with the center of the particle's circular trajectory, such that the wave vector $\mathbf{k}$ is parameterized by the polar θ and azimuthal φ spherical angles, $\mathbf{k} = \left(|\mathbf{k}_\perp|\cos\varphi,\ |\mathbf{k}_\perp|\sin\varphi,\ k_\parallel\right)$, $|\mathbf{k}_\perp| = k_0\sin\theta$, $k_\parallel = k_0\cos\theta$, we obtain:

$$\begin{aligned}
y_{\mathbf{k}1}(t) &= y_{\mathbf{k}1}(t,0) = -\frac{iqR\cos\theta}{2\pi\sqrt{\hbar ck_0}}Y_{\mathbf{k}}(\varphi) \\
&\times \int_{\tau_i}^{\tau_i+\Omega t}\exp\left[i\left(\frac{ck_0}{\Omega}\tau - |\mathbf{k}_\perp|\,R\sin\tau\right)\right]\cos\tau d\tau \, , \\
y_{\mathbf{k}2}(t) &= y_{\mathbf{k}2}(t,0) = -\frac{iqR}{2\pi\sqrt{\hbar ck_0}}Y_{\mathbf{k}}(\varphi) \\
&\times \int_{\tau_i}^{\tau_i+\Omega t}\exp\left[i\left(\frac{ck_0}{\Omega}\tau - |\mathbf{k}_\perp|\,R\sin\tau\right)\right]\sin\tau d\tau \, , \\
Y_{\mathbf{k}}(\varphi) &= \exp\left[i\frac{ck_0}{\Omega}(\varphi - \pi/2)\right] \, .
\end{aligned} \tag{2.4}$$

To obtain these integrals, we performed a changed of variables $\Omega t' = \tau - \tau_i$, $\tau_i = \pi/2 - \varphi$, and used the following representation of the polarization vectors $\boldsymbol{\epsilon}_{\mathbf{k}1} = (\cos\varphi\cos\theta, \sin\varphi\cos\theta, -\sin\theta)$, $\boldsymbol{\epsilon}_{\mathbf{k}2} = (-\sin\varphi, \cos\varphi, 0)$. Next, using the plane-wave expansions of the Bessel functions $J_n(z)$ given by Eqs. (9.56) in [8] and performing additional manipulations as described in [10], the total electromagnetic energy (1.14) radiated by the point charge has the form [10]

$$\begin{aligned}
W(t) &= \frac{q^2\Omega^2}{2\pi}\sum_{n=-\infty}^{+\infty}\int_0^\infty dk_0\left(\frac{2}{ck_0 - n\Omega}\right)^2\sin^2\left(\frac{ck_0 - n\Omega}{2}t\right) \\
&\times \int_0^\pi \sin\theta\left[n^2J_n^2(|\mathbf{k}_\perp|\,R)\cot^2\theta + k_0^2R^2J_n'^2(|\mathbf{k}_\perp|\,R)\right]d\theta \, .
\end{aligned} \tag{2.5}$$

Differentiating the energy (2.5) we obtain the rate (1.15) at which energy is emitted by the source:

$$\begin{aligned}
w(t) &= \frac{q^2\Omega^2}{2\pi}\sum_{n=-\infty}^{+\infty}\int_0^\infty dk_0\frac{2\sin[(ck_0 - n\Omega)t]}{ck_0 - n\Omega} \\
&\times \int_0^\pi \sin\theta\left[n^2J_n^2(|\mathbf{k}_\perp|\,R)\cot^2\theta + k_0^2R^2J_n'^2(|\mathbf{k}_\perp|\,R)\right]d\theta \, .
\end{aligned} \tag{2.6}$$

The above expression is a generalization of the well-known Schott's formula [14], owing to the explicit dependence on time. In the semiclassical formulation, the quantum transition interval $\Delta t = t - t_{\text{in}} = t$

determines the interval where radiation is formed; in other words, Δt is the radiation formation interval. This feature is absent in the classical theory. However, in the limit where the interval $t \to \infty$ we obtain Schott's classical formula [14]:

$$\begin{aligned} w &= \lim_{t\to\infty} w(t) = \frac{q^2\Omega^2}{2\pi} \sum_{n=-\infty}^{+\infty} n^2 \\ &\times \int_0^\pi \sin\theta \left[J_n^2\left(\frac{n\Omega R}{c}\sin\theta\right)\cot^2\theta + \left(\frac{\Omega R}{c}\right)^2 J_n'^2\left(\frac{n\Omega R}{c}\sin\theta\right)\right] d\theta\,. \end{aligned} \tag{2.7}$$

2.2. Point charge in a constant and homogeneous electric field

Consider the point charge accelerated by a constant and homogeneous electric field, $\mathbf{E} = (0, 0, E)$. The current density describing the point charge is given by Eq. (2.1). Its trajectory and velocity with initial data $\underline{\mathbf{r}} = \mathbf{r}(0) = (\underline{x}, \underline{y}, \underline{z})$, $\underline{\mathbf{v}} = \mathbf{v}(0) = (\underline{v}_x, \underline{v}_y, \underline{v}_z)$ read:

$$\begin{aligned} &\mathbf{r}_\perp(t) = \tilde{\mathbf{r}}_\perp + \frac{\underline{\mathbf{u}}_\perp}{\varepsilon}\operatorname{arcsinh}\left(\frac{\varepsilon t + \underline{u}_\parallel/c}{\varrho}\right), \quad \boldsymbol{\beta}_\perp(t) = \frac{\underline{\mathbf{u}}_\perp/c}{\sqrt{\varrho^2 + \left(\varepsilon t + \underline{u}_\parallel/c\right)^2}}\,, \\ &r_\parallel(t) = \underline{r}_\parallel + \frac{c}{\varepsilon}\sqrt{\varrho^2 + \left(\varepsilon t + \underline{u}_\parallel/c\right)^2}, \quad \beta_\parallel(t) = \frac{\varepsilon t + \underline{u}_\parallel/c}{\sqrt{\varrho^2 + \left(\varepsilon t + \underline{u}_\parallel/c\right)^2}}\,, \\ &\varepsilon = \frac{qE}{mc}, \quad \varrho = \frac{\sqrt{m^2c^2 + \underline{\mathbf{P}}_\perp^2}}{mc}, \quad \underline{\mathbf{P}}_\perp = m\underline{\mathbf{u}}_\perp, \quad \underline{\mathbf{u}} = \frac{c\underline{\boldsymbol{\beta}}}{\sqrt{1-\underline{\boldsymbol{\beta}}^2}}\,. \end{aligned} \tag{2.8}$$

Here, $\tilde{\mathbf{r}}_\perp = \underline{\mathbf{r}}_\perp - (\underline{\mathbf{u}}_\perp/\varepsilon)\operatorname{arcsinh}\left(\underline{u}_\parallel/\varrho c\right)$ and the indexes "$\perp$", "$\parallel$" here label components "perpendicular", "parallel" to the external field, respectively. Using the above solutions, we substitute t' by η'

$$t' = -\frac{\underline{u}_\parallel}{\varepsilon c} + \frac{\varrho}{\varepsilon}\sinh\eta'\,, \tag{2.9}$$

and introduce three auxiliary variables z, ξ, ν,

$$z = \frac{c\varrho}{\varepsilon}|\mathbf{k}_\perp|\,, \quad \xi = \frac{1}{2}\ln\left(\frac{|\mathbf{k}| + k_\parallel}{|\mathbf{k}| - k_\parallel}\right), \quad \nu = \frac{\mathbf{k}_\perp\underline{\mathbf{u}}_\perp}{\varepsilon}, \quad |\mathbf{k}_\perp| \neq 0\,, \tag{2.10}$$

to rewrite the phase $\Phi(t')$ (2.3) as follows

$$\Phi(\eta') = z\sinh(\eta' - \xi) - \nu\eta' + \tilde{\mathcal{C}}\,, \tag{2.11}$$

where $\tilde{\mathcal{C}}$ is a phase. As a result, the complex function (2.3) admits the representation

$$\begin{aligned} y_{\mathbf{k}\lambda}(t, t_{\text{in}}) &= -i\frac{qc}{2\pi}\frac{e^{-ik_0ct_{\text{in}}}e^{i\tilde{\mathcal{C}}}}{\varepsilon\sqrt{\hbar\omega}}\boldsymbol{\epsilon}^*_{\mathbf{k}\lambda}\mathbf{I}_\nu(t, t_{\text{in}})\,, \\ \mathbf{I}_\nu(t, t_{\text{in}}) &= \left(\frac{\underline{\mathbf{u}}_\perp}{c}I_\nu^{(1)}(t, t_{\text{in}}), \varrho I_\nu^{(2)}(t, t_{\text{in}})\right)\,, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} &I_\nu^{(1)}(t, t_{\text{in}}) = \int_{\eta_{\text{in}}}^{\eta} e^{i\left[z\sinh\left(\eta' - \xi\right) - \nu\eta'\right]}d\eta'\,, \\ &I_\nu^{(2)}(t, t_{\text{in}}) = \int_{\eta_{\text{in}}}^{\eta} e^{i\left[z\sinh\left(\eta' - \xi\right) - \nu\eta'\right]}\sinh\eta' d\eta'\,, \\ &\eta = \eta(t) = \operatorname{arcsinh}\left(\varepsilon t/\varrho + \underline{u}_\parallel/\varrho c\right), \quad \eta_{\text{in}} \equiv \eta(t_{\text{in}})\,. \end{aligned} \tag{2.13}$$

By performing a supplementary change of variable $u' = \eta' - \xi$ we may express the above integrals in terms of an "incomplete" Macdonald function,

$$K_{i\nu}\left(z;t,t_{\text{in}}\right) = \frac{e^{-\pi\nu/2}}{2}\int_{u_{\text{in}}}^{u} e^{i\phi\left(u'\right)}du', \quad \phi\left(u'\right) = z\sinh u' - \nu u' \,. \tag{2.14}$$

Finally, calculating the modulus square of Eq. (2.12) and summing the result over the photon polarizations with the aid of the identities (1.5), the total electromagnetic energy radiated by the particle (1.14) takes the form [11]:

$$\begin{aligned} W\left(t,t_{\text{in}}\right) &= \left(\frac{qc}{\varepsilon\pi}\right)^2 \int e^{\pi\nu}\left\{\left[\left(1-\frac{\nu^2}{z^2}\right)\varrho^2 - 1\right]\left|K_{i\nu}\left(z;t,t_{\text{in}}\right)\right|^2 \right. \\ &+ \left. \varrho^2\left|S_{i\nu}\left(z;t,t_{\text{in}}\right)\right|^2\right\} d\mathbf{k}\,, \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} S_{i\nu}\left(z;t,t_{\text{in}}\right) &= K'_{i\nu}\left(z;t,t_{\text{in}}\right) - \frac{1}{z}\frac{k_{\parallel}}{|\mathbf{k}|}\dot{K}_{i\nu}\left(z;t,t_{\text{in}}\right)\,, \\ K'_{i\nu}\left(z;t,t_{\text{in}}\right) &= \partial_z K_{i\nu}\left(z;t,t_{\text{in}}\right)\,, \end{aligned}$$

and

$$\begin{aligned} \dot{K}_{i\nu}\left(z;t,t_{\text{in}}\right) &= \partial_\xi K_{i\nu}\left(z;t,t_{\text{in}}\right) \\ &= \begin{cases} e^{-\pi\nu/2}\frac{e^{i\left(z\sinh u_{\text{in}}-\nu u_{\text{in}}\right)}-e^{i\left(z\sinh u-\nu u\right)}}{2} & \text{if } -\infty < t_{\text{in}} < t < +\infty\,, \\ 0 & \text{if } t = -t_{\text{in}} = +\infty\,. \end{cases} \end{aligned} \,.$$

This compact expression corresponds to a generalization of the classical differential energy due to its dependence on time; cf. Eq. (16) in Ref. [15]. In the limit $t \to \infty$, $t_{\text{in}} \to -\infty$, we recover the result obtained by Nikishov and Ritus in classical theory [15]:

$$W = \lim_{\Delta t\to\infty} W\left(t,t_{\text{in}}\right) = \left(\frac{qc}{\varepsilon\pi}\right)^2\int e^{\pi\nu}\left\{\left[\left(1-\frac{\nu^2}{z^2}\right)\varrho^2-1\right]K_{i\nu}^2\left(z\right) + \varrho^2 K_{i\nu}'^2\left(z\right)\right\}d\mathbf{k}\,.$$

It remains to discuss the energy rate emitted by the accelerated particle. Differentiating Eq. (2.15) with respect to time we find:

$$\begin{aligned} w\left(t,t_{\text{in}}\right) &= \left(\frac{qc}{\pi}\right)^2\frac{1}{\varepsilon\varrho}\int\frac{e^{\pi\nu/2}}{\cosh\eta}\left\{\left[\left(1-\frac{\nu^2}{z^2}\right)\varrho^2-1\right]\text{Re}\left[e^{i\phi(u)}K_{i\nu}^*\left(z;t,t_{\text{in}}\right)\right]\right. \\ &+ \left.\frac{|\mathbf{k}_\perp|}{|\mathbf{k}|}\left(\sinh\eta - \frac{\nu}{z}\frac{k_\parallel}{|\mathbf{k}_\perp|}\right)\varrho^2\text{Re}\left[ie^{i\phi(u)}S_{i\nu}^*\left(z;t,t_{\text{in}}\right)\right]\right\}d\mathbf{k}\,. \end{aligned} \tag{2.16}$$

This equation is our main result. It expresses the electromagnetic energy radiated by the particle within the quantum transition interval $\Delta t = t - t_{\text{in}}$. Similarly to the total energy (2.15), Eq. (2.16) corresponds to a generalization of the classical energy rate radiated by the particle accelerated by the electric field. The computation of the rate simplifies considerably if we set $t_{\text{in}} \to -\infty$ and restrict ourselves to the case where the particle is subjected to the initial condition $\underline{\mathbf{v}}_\perp = \mathbf{0}$. In this case, $\varrho = 1$, $\nu = 0$, and the energy rate (2.16) assumes the form

$$\begin{aligned} &w\left(t\right)\big|_{\underline{\mathbf{v}}_\perp=\mathbf{0}} = \frac{2\left(qc\right)^2}{\pi\varepsilon}\tanh\eta \\ &\times\int_0^\infty d\left|\mathbf{k}_\perp\right|\mathbf{k}_\perp^2\int\frac{\cos\left(z\sinh u\right)\text{Im}K_0'\left(z;t\right)-\sin\left(z\sinh u\right)\text{Re}K_0'\left(z;t\right)}{\sqrt{\mathbf{k}_\perp^2+k_\parallel^2}}dk_\parallel\,. \end{aligned} \tag{2.17}$$

where $K_0'\left(z;t\right) = K_0'\left(z;t,-\infty\right)$. In the limit $t\to+\infty$, the integral

$$K_{i\nu}\left(z;t\right) \equiv K_{i\nu}\left(z;t,-\infty\right) = \frac{e^{-\pi\nu/2}}{2}\int_{-\infty}^{u}e^{i\phi\left(u'\right)}du'\,. \tag{2.18}$$

becomes the Macdonald function and $\mathrm{Im}K_0'(z;+\infty)=0$, $\mathrm{Re}K_0'(z;+\infty)=K_0'(z)$. Thus, performing a supplementary change of variables and using the identity $K_0'(z)=-K_1(z)$ we finally obtain

$$w|_{\mathbf{v}_\perp=\mathbf{0}}=\lim_{t\to\infty} w(t)|_{\mathbf{v}_\perp=\mathbf{0}}=\frac{q^2\varepsilon^2}{c}\int_0^\infty K_1(z)\,z^2dz=2\frac{q^2}{c^3}a^2\,,\quad a=\frac{qE}{m}\,. \tag{2.19}$$

Except by a factor of 1/3, this result coincides with Larmor's formula for the total energy rate radiated by a uniformly accelerated charged particle [7, 16]. The absence of this factor was also pointed out in the framework of the classical theory by Nikishov and Ritus in Ref. [15].

3. Conclusion

In this work we addressed the problem of the electromagnetic radiation produced by charge distributions in a semiclassical approach, in which the radiation field is quantum while current densities–sources of radiation–are regarded classically. In this formulation, pertinent electromagnetic quantities, such as energies and energy rates radiated by currents, are calculated with the aid of transition probabilities between states with a well-defined number of photons. Assuming the vacuum as the initial state, we calculated time-dependent one-photon, multi-photon, total electromagnetic energies and the rate at which the radiation is emitted from the source. We discovered that our formulas for the total energy and rate are compatible with the corresponding classical results in the limit where the quantum transition interval tends to infinity. To illustrate the use of the semiclassical approach, we summarized our recent results on the synchroton radiation [10] and the radiation by a charged particle in rectilinear accelerated motion [11]. We conclude this work by emphasizing that the semiclassical approach offers an alternative description of physical systems interacting with background fields. Despite being an approximation compared to QED, the semiclassical formulation exactly incorporates the quantum character of the electromagnetic field. For this reason, this theory allows extracting information about electromagnetic properties stemming from the interaction between radiation and matter beyond the reach of classical electrodynamics.

Список литературы/References

1. Heitler, W. *The quantum theory of radiation.* Courier Corporation, 1984.
2. Schweber, Silvan S. *An introduction to relativistic quantum field theory.* Courier Corporation, 2013.
3. N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, 3rd Ed. (John Wiley & Sons, New York 1980).
4. Poynting, J. H. "On the transfer of energy in the electromagnetic field."Proceedings of the Royal Society of London 36.228-231 (1883): 186-187.
5. Heaviside, O. "The induction of currents in cores."Electrician 13, 1884, 133.
6. L. D. Landau and E. M. Lifshitz, *The classical theory of fields* (Pergamon Press, Oxford 1971).
7. J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, New York 1999).
8. A. A. Sokolov, I. M. Ternov, *Synchrotron Radiation* (Academic Verlag, Berlin 1968); *Radiation from relativistic electrons*(American Institute of Physics, New York 1986).
9. J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York 1941).
10. V. G. Bagrov, D. M. Gitman, A. A. Shishmarev and A. J. D. Farias, Quantum states of an electromagnetic field interacting with a classical current and their applications to radiation problems, *J. Synchrotron Rad.* **27**, 902 (2020).
11. T. C. Adorno, A. I. Breev, A. J. D. Farias, and D. M. Gitman, Electromagnetic radiation of accelerated charged particle in the framework of a semiclassical approach, *Ann. Phys.* (Berlin) **2023**, 2300118, 2023.

12. A. A. Shishmarev, A. D. Levin, V. G. Bagrov, D. M. Gitman, Semiclassical Description of Undulator Radiation, *JETPh* **132**, (2) 247 (2021).
13. W. Greiner, J. Reinhardt, *Field Quantization* (Springer, Berlin 1996).
14. G. A. Schott, On the radiation from moving systems of electrons, and on the spectrum of canal rays, *Phil. Mag.* **13**, 657 (1907).
15. A. I. Nikishov and V. I. Ritus, Radiation spectrum of an electron moving in a constant electric field, *Zh. Eksp. Teor. Fiz.* **56**, 2035 (1969) [*Sov. Phys. JETP* **29**, 1093 (1969)].
16. J. Larmor, On the theory of the magnetic influence on spectra; and on the radiation from moving ions, *Phil. Mag.* **44**, No. 271, 503 (1897).

Авторы

Тьяго Карлос Адорно де Фрейтас, к.ф.-м.н., доцент, Факультет физики, Сианьский университет Цзяотун-Ливерпуль, 111 Ren'ai Road, Сучжоу, район науки и образования озера Душу, промышленный парк Сучжоу, Сучжоу, 215123, Китай.
E-mail: Tiago.Adorno@xjtlu.edu.cn

Александр Игоревич Бреев, к.ф.-м.н., доцент кафедры теоретической физики ФФ, Томский государственный университет, пр. Ленина 36, Томск, 634050, Россия.
E-mail: breev@mail.tsu.ru

Дмитрий Максимович Гитман, к.ф.-м.н., профессор, Физический институт им. П. Н. Лебедева, Ленинский проспект 53, 119991, Москва, Россия; Институт физики, Университет Сан-Паулу, улица Матан, 1371, Сан-Паулу 05508-090, Бразилия; Томский государственный университет, пр. Ленина 36, Томск, 634050, Россия.
E-mail: gitman@if.usp.br

Просьба ссылаться на эту статью следующим образом:

Адорно Т. К., Бреев А. И., Гитман Д. М. Квазиклассическое описание электромагнитного излучения ускоренных зарядов. *Пространство, время и фундаментальные взаимодействия.* 2023. № 3-4. С. 12–20.

Authors

Tiago Carlos Adorno de Freitas, Ph.D., Assistant Professor, Department of Physics, Xi'an Jiaotong-Liverpool University, 111 Ren'ai Road, Suzhou Dushu Lake Science and Education Innovation District, Suzhou Industrial Park, Suzhou, 215123, P. R. China.
E-mail: Tiago.Adorno@xjtlu.edu.cn

Alexander Igorevich Breev, Ph.D., Associate Professor, Department of Physics Tomsk State University Lenin Ave. 36, Tomsk 634050, Russia.
E-mail: breev@mail.tsu.ru

Dmitri Maximovich Gitman, Ph.D., Full Professor, P. N. Lebedev Physical Institute 53 Leninskiy prospekt, 119991 Moscow, Russia; Instituto de Física, Universidade de São Paulo, Rua do Matão, 1371, São Paulo 05508-090, Brazil; Department of Physics Tomsk State University Lenin Ave. 36, Tomsk 634050, Russia.
E-mail: gitman@if.usp.br

Please cite this article in English as:

Adorno T. C., Breev A. I., Gitman D. M. Semiclassical description of the electromagnetic radiation by accelerated charged distributions. *Space, Time and Fundamental Interactions*, 2023, no. 3-4, pp. 12–20.