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# TO PROFESSOR A.Z. PETROV ON HIS 90 ANNIVERSARY

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*Lecture of Professor Graham Hall (Aberdeen, Scotland)*
*at the opening ceremony of the XII Petrov School held on June 26, 2000,*
*in the Museum of the History of Kazan State University*

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Alexei Zinovievich Petrov was born in the village of Koshki, Samara on October 28 (October 15 in the old style calendar) 1910. He was the eleventh of the twelve children of the village pastor. Various catastrophes in his early years led to his adoption by his aunt Ekaterina Vasilievna Petrova and it is from her that he got his surname.

His early education in the town of Melex did not immediately suggest his illustrious future; he fell behind at mathematics! However the damage to his pride caused him to fight back and this experience together with a later one (when he discovered a higher university textbook on mathematics whilst browsing in a bookshop) secured his future in mathematics. He entered the department of physics and mathematics in the University of Kazan in 1932 and turned out to be a brilliant student. Although his later research was interrupted by the Second World War (in which Petrov played a significant role in the defence of Moscow and was badly wounded) he finally completed his PhD in Kazan under the guidance of Professor Shirokov.

The content of Petrov's doctoral work was to have a profound effect on Einstein's general theory of relativity. And since Einstein's theory is a geometrical theory of gravitation which rejects the a priori assumption of Euclidean geometry usually made in classical physics, it is fitting that Petrov's work should be accomplished in Kazan. For it was in that same institution that Nicolai Ivanovich Lobachevsky (1793–1856) carried out his famous pioneering work on non-Euclidean geometry. Petrov's work led to his appointment to the chair in Kazan in 1956 and he built a large and successful group around him. Petrov acquired a reputation for being immensely hardworking and also for his ability to focus attention on the important points in an argument and not to be diverted elsewhere. This trait shows itself in the clarity (and conciseness) of his scientific writings. Much of his most important work is described in detail in his book "Einstein Spaces" and it is, perhaps, through this book that non-Russian speaking relativists get their first direct experience of Petrov's work.

Although Petrov's research covered many topics, it is his famous classification ("The Petrov Classification") of gravitational fields that he will probably be best remembered for amongst mathematicians and physicists. This work was originally an algebraic classification of the curvature tensor of a space-time which is also an Einstein space (in the sense used by differential geometers). Since the details differ little from the situation when the curvature tensor represents a vacuum space-time, this work was inevitably applied to the Weyl conformal tensor of any space-time since this tensor has the same algebraic symmetries as a vacuum curvature tensor. The idea is to use the algebraic symmetries of the Weyl tensor to write it as a 6x6 real matrix and then to study the eigenvector-eigenvalue problem for this matrix with respect to a rank 6 quadratic form of signature $(+++---)$. Petrov showed that 3 main algebraic types arise (now labelled Petrov types $I, II$ and $III$) together with a degeneracy of each of the types $I$ and $II$, labelled $D$ and $N$, respectively. With the trivial case (when the Weyl tensor vanishes at the point in question) labelled type $O$ one has the complete set of Petrov types $I, D, II, N, III$ and $O$ for the algebraic nature of the Weyl tensor at each point of space-time.

Petrov's work was developed in two main directions, especially by Penrose and Pirani in England, Bel and Lichnerowicz in France, Debever and Geheniau in Belgium and Ehlers, Sachs and Kundt in Germany, but also by many others too numerous to mention. One of these developments was essentially physical and was analogous to a similar (but simpler) algebraic classification of the Maxwell tensor representing an electromagnetic field. It attempted to link Petrov's classification with the existence of gravitational radiation. Perhaps the early attempts at this were too simplistic and a more realistic view is that in the general case a space-time is of Petrov type and that the set of space-times of Petrov type contain interesting examples of metrics suggesting gravitational radiation. The other development of Petrov's work was geometrical and one of the most elegant and useful aspects of this work was the discovery of certain special null directions associated with a point of space-time and which depended on the Petrov type at that point. This is, at least, for the present author the most beautiful of the consequences of Petrov's work and was achieved mainly by Bel, Debever and Sachs using tensor techniques and by Penrose who reformulated Petrov's ideas in an elegant spinor formalism.

Perhaps the most practically useful consequence of the Petrov classification has been in the quest for exact solutions of Einstein's field equations. In the early days of general relativity many of the exact solutions were found after imposing certain symmetries on the model in question (and, indeed, one of the most comprehensive discussions of space-times admitting metric symmetries is to be found in Petrov's "Einstein Spaces"). After Petrov's work one could impose, in addition to symmetries, a particular fixed Petrov type at each space-time point. This technique, sometimes allied to the rather useful "Newman-Penrose formalism", enabled many more exact solutions of Einstein's field equations to be discovered.

At the end of 1969 Petrov was elected an Academician of the Ukrainian Academy of Sciences and moved to Kiev. In 1972 he received the Lenin award for his achievements. However many years of hard work had taken its toll on his health and on May 9, 1972, Petrov died in hospital when, following surgery, a blood clot developed in his heart.

Let me now add some personal notes on Professor Petrov's work. I first came across the Petrov classification as a Ph.D student at the University of Newcastle-upon-Tyne in England in the early 1970s. Although I was working on Einstein's theory, I had, in fact, graduated in mathematics from Newcastle and was still, in many ways, a mathematician at heart. This classification of gravitational fields appealed greatly to me both as on object of beauty and as an extremely useful tool for both theoretical and practical general relativity. Unfortunately, I never met Professor Petrov. However, I have had the great pleasure of meeting several of his former students during the (now, many) occasions I have visited Kazan and I wish to thank them for their friendship and hospitality.

I have added a brief appendix which attempts to describe the essential mathematical idea behind Petrov's classification. I should also add that Petrov's work has recently been chosen for reprinting in the journal General Relativity and Gravitation (in English) and has already appeared. It is referred to in the appendix.

**Appendix.**

## The Petrov Classification of Gravitational Fields

In the renaissance of general relativity which took place from about the middle of the 1950s, the classification of gravitational fields developed by Professor A.Z. Petrov was of outstanding importance. This work, first published by Petrov in 1951 [1] and refined in the famous paper of 1954 [2], was largely instrumental in providing the basis for the mathematical development of exact solutions of Einstein's field equations and for the physical interpretation of general relativity theory. The details of Petrov's original work is, fortunately, readily available, not only in his well-known textbook "Einstein Spaces" [3] but also through the recent reprinting, in English, of his 1954 paper [4].

Petrov's original classification was applied to the Riemann tensor of a space-time manifold M of dimension four which carried a Lorentz metric g and which was assumed to satisfy the Einstein space condition that the Ricci tensor was proportional to g. This, of course, included the important cases of vacuum space-times.

Nowadays, Petrov's classification is usually applied to the Weyl (conformal curvature) tensor where the techniques used are essentially identical to those used by Petrov. Although a "modern" version of Petrov's classification will be briefly presented here, the approach and techniques are as in the original paper.

To describe the Petrov classification, let $C$ be the Weyl tensor with components $C_{abcd}$. Then one has the familiar properties of $C$:

$$C_{abcd} = -\,C_{bacd} = -C_{abdc} \tag{1}$$

$$C_{abcd} = C_{cdab} \tag{2}$$

$$C_{a[bcd]} = 0, \quad C^{a}_{bcd} = 0, \tag{3}$$

$$^{*}C_{abcd} = C^{*}_{abcd} \quad (\Rightarrow {}^{*}C^{*}_{abcd} = -C_{abcd}), \tag{4}$$

where square brackets denote the usual skew-symmetrisation and * denotes the duality (Hodge) operator. Now consider the algebraic eigenbivector — eigenvalue problem for $C$ at some point $m \in M$ given by

$$C_{abcd}F^{cd} = \lambda F_{ab} \tag{5}$$

where $F_{ab} = -F_{ba}$ are the components of a complex bivector $F$ at $m$ and $\lambda \in C$. Here, Petrov has taken advantage of the properties (1) of $C$ to regard it as a $6 \times 6$ matrix representing in an obvious way a linear map $f$ from the 6-dimensional complex vector space of complex bivectors at $m$ to itself.

His classification is then an algebraic classification of this linear map through the matrix $C_{ab}$ $(1 \leqslant A, B \leqslant 6)$, where Petrov linked his 6-dimensional indices A and B to the skew-symmetric bivector index-pairs according to the scheme $1-14$, $2-24$, $3-34$, $4-23$, $5-31$, $6-12$.

Now let $F \in V$. Then $F$ is called *self dual* if $\overset{*}{F} = -iF$ and *anti-self dual* if $\overset{*}{F} = iF$. If $S^{+}$ and $S^{-}$ denote, respectively, the subsets of self dual and anti-self dual members of $V$ then they are subspaces of $V$ and, in fact, $V$ is their direct sum, $V = S^{+} \oplus S^{-}$. Further, the linear map $f$ described above and constructed from $C$ has the property that $f$ maps $S^{+}$ into itself and $S^{-}$ into itself. Thus the map $f$ is completely described by its *restrictions* $f_1 = S^{+} \rightarrow S^{+}$ and $f_2 = S^{-} \rightarrow S^{-}$.

Petrov's original problem of algebraically classifying the linear map $f$ at $m \in M$ by studying its possible Jordan canonical forms is now greatly simplified since it is easy to check that $f_1$ and $f_2$ have identical Jordan forms as linear maps on $S^{+}$ and $S^{-}$, respectively, that is, on $\mathbf{C}^3$. This common Jordan type can then be defined to be the algebraic (Petrov) type of $C$ at $m$.

There are, then, three possible algebraic types for $C$ at $m$ corresponding to the Jordan types which, in Segre notation, are 111, 21 and 3 (assuming $C(m) \neq 0$). These are, at $m$, the three types of gravitational field announced by Petrov over fifty years ago. It is customary now to label these types so as to include the two possible algebraic degeneracies in each of the first two types. Thus the modern version of this classification is

$$\begin{aligned} &\text{Petrov Type I} && (\{111\} - \text{no degeneracies}) \\ &\text{Petrov Type D} && (\{1(11)\}) \\ &\text{Petrov Type II} && (\{21\} - \text{no degeneracies}) \\ &\text{Petrov Type N} && (\{(21)\}) \\ &\text{Petrov Type III} && (\{3\}). \end{aligned} \tag{6}$$

In the types $N$ and $III$ the tracefree property in (1) shows that all eigenvalues of $f$ (or $f_1$ and $f_2$) are zero. One normally adds a Petrov type $O$ to cover the possibility of $C(m) = 0$. However, it must be remembered that Petrov's classification is *pointwise* and the Petrov type can be expected to vary from point to point on $M$ (subject, of course, to continuity restrictions).

Since Petrov's original papers appeared, several other approaches to his classification have been proposed. Perhaps the best known of these are the spinor classification of Penrose [5] and the approach centred on "principal null directions" due mainly to Bel [6] (but augmented by Debever [7] and by Sachs, Ehlers and Kundt [8], [9]). In these approaches one looks for solutions for a null vector $k$ at $m$ such that

Such a solution $k$ is said to span a principal null direction of $C$. It turns out that there is at least one and at most four independent such solutions and, in fact, four in the type $I$ case, three for type $II$, two for each of the types $D$ and $III$ (and which be distinguished by a more detailed examination of (4)) and one for type $N$. These approaches greatly facilitated the process of calculation using Petrov's classification.

As well as being a beautiful piece of mathematics, Petrov's classification led to much progress in the physics of general relativity. In particular, by being able to study Einstein's field equations for a particular Petrov type, many more exact solutions of Einstein's equations were found. Also, by use of the elegant geometrical concepts involved (and analogies with Maxwell's electromagnetic theory) the physical interpretation of general relativity was greatly clarified. Petrov's classification scheme is, indeed, one of the major developments in Einstein's general relativity theory.

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