ГРАВИТАЦИЯ, КОСМОЛОГИЯ И ФУНДАМЕНТАЛЬНЫЕ ПОЛЯ

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H-ПРОСТРАНСТВА (H_{41}, G) ТИПА {41}: ПРОЕКТИВНО-ГРУППОВЫЕ СВОЙСТВА

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В данной работе исследуются пятимерные h-пространства (H_{41}, g) типа {41} [4]. Находятся необходимые и достаточные условия, при которых (H_{41}, g) является пространством постоянной кривизны. Определяется общее решение уравнения Эйзнхарта в h-пространстве (H_{41}, g) непостоянной кривизны. Устанавливаются условия существования негомотетического проективного движения в (H_{41}, g) и описывается структура негомотетической проективной алгебры Ли в h-пространстве (H_{41}, g) типа {41}.

Ключевые слова: пятимерное псевдориманово многообразие, h-пространство типа {41}, уравнение Эйзенхарта, проективная алгебра Ли.

H-SPACES (H_{41}, G) OF TYPE {41}: PROJECTIVE-GROUP PROPERTIES

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In this paper we study five-dimensional h-spaces (H_{41}, g) of type {41} [4]. Necessary and sufficient conditions for (H_{41}, g) to be a space of constant curvature are found. The general solution of the Eisnhart equation in h-space (H_{41}, g) of non-constant curvature is determined. We establish conditions for the existence of a non-homothetic projective motion in (H_{41}, g) and describe the structure of a non-homothetic projective Lie algebra in h-space (H_{41}, g) of type {41}.

Keywords: five-dimensional pseudo-Riemannian manifold, h-space of type {41}, Eisenhart equation, projective Lie algebra.

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Introduction

A vector field X on a five-dimensional pseudo-Riemannian manifold (M, g) with a projective structure Π is called an *infinitesimal projective transformation*, or a *projective motion* if the local 1-parameter group of local transformations, which is generated by this field in a neighbourhood of each point $x \in M$, consists of (local) projective transformations, that is, automorphisms of the projective structure Π.

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The necessary and sufficient condition that $X = \xi^i \partial_i$ be a projective motion in a pseudo–Riemannian manifold (M, g) is

$$(L_X g_{ij})_{,k} = 2g_{ij}\varphi_{,k} + g_{ik}\varphi_{,j} + g_{jk}\varphi_{,i}, \tag{1}$$

where φ is a function of x^i, which we shall call a *defining function* of a projective motion X.

The equation (1) can be written in the form of two relations:

$$L_X g_{ij} \equiv \xi_{i,j} + \xi_{j,i} = h_{ij} \tag{2}$$

(the *generalized Killing equation*) and

$$h_{ij,k} = 2g_{ij}\varphi_{,k} + g_{ik}\varphi_{,j} + g_{jk}\varphi_{,i} \tag{3}$$

(the *Eisenhart equation*). If $\varphi = \text{const}$, that is, $\text{div}X = \text{const}$, then the vector field X preserves the affine connection and hence it is an infinitesimal affine transformation, or an *affine motion*.

An affine motion X is an infinitesimal homothety, or a *homothetic motion* when $h_{ij} = const \cdot g_{ij}$ and an infinitesimal isometry, or an *isometric motion* when $h_{ij} = 0$ [1].

After making a change of variables

$$h_{ij} = a_{ij} + 2\varphi g_{ij},$$

where a_{ij} is a symmetric bilinear form with the same Segre characteristic χ (that is of the same type χ) as h_{ij}, the equation (3) becomes

$$a_{ij,k} = g_{ik}\varphi_{,j} + g_{jk}\varphi_{,i}. \tag{4}$$

We call g_{ij} an *h-metric of type* χ and we call (M, g) an *h-space of type* χ [2].

Given the type of tensor h_{ij}, one can find solutions of the Eisenhart equation (3) and then of the Killing equation (2).

In a canonical skew-normal frame ([3], p. 97) $\{Y_l\}$ on $V \subseteq M$ the equation (4) takes the form

$$d\bar{a}_{pq} + \sum_{h=1}^{n} e_h(\bar{a}_{hq}\omega_{p\tilde{h}} + \bar{a}_{ph}\omega_{q\tilde{h}}) = (Y_q\varphi)\theta_p + (Y_p\varphi)\theta_q,$$

where θ_h is the canonical 1-form which is conjugate to Y_h, $\omega_{pq} = -\omega_{qp}$ is the connection 1-form, and $p, q, r = 1, \ldots, 5$.

In [4] h-spaces (H_{41}, g) of type $\{41\}$ were found, and necessary and sufficient conditions for the existence of projective motion of type $\{41\}$ were obtained. To calculate the maximal projective Lie algebra in (H_{41}, g), it is necessary to obtain a general solution of the Eisenhart equation (3) in h-space (H_{41}, g). To solve this problem, one needs to study the integrability conditions (25) for the Eisenhart equation, which contain the curvature form Ω_{ij}. In this case, spaces of constant curvature should be excluded, the structure of the projective group of which is well known [2].

The outline of the article is as follows. Basic definitions and formulas are given in Section 1. The curvature structure of the h-space (H_{41}, g) is defined in Section 2. In Section 3 we derive the necessary and sufficient conditions that h-space (H_{41}, g) be a space of constant curvature. In Section 4 we discuss the integrability conditions of the Eisenhart equation in (H_{41}, g) and obtain an important characteristic of its solutions in an h-space (H_{41}, g) of non-constant curvature. In Section 5 we establish necessary and sufficient conditions for the existence of a non-homothetic projective motion in h-space (H_{41}, g) of type $\{41\}$, and determine the structure of a non-homothetic projective Lie algebra in (H_{41}, g).

1. Computing curvature of (H_{41}, g).

In the paper [4] a canonical skew-normal frame $(Y_i) = (\underset{i}{\xi}^j \partial_j)$ has been defined with the following components in the appropriate coordinates:

$$\underset{1}{\xi}^1 = \underset{2}{\xi}^2 = \underset{3}{\xi}^3 = \frac{1}{(f_2 - f_1)^{1/2}}, \quad \underset{2}{\xi}^1 = \frac{1}{2(f_2 - f_1)^{3/2}},$$

$$\underset{3}{\xi}^1 = \frac{3}{8(f_2-f_1)^{5/2}}, \quad \underset{3}{\xi}^2 = \frac{1}{2(f_2-f_1)^{3/2}}, \quad \underset{4}{\xi}^1 = \frac{5}{16(f_2-f_1)^{7/2}}, \tag{5}$$

$$\underset{4}{\xi}^2 = \frac{1}{8(f_2-f_1)^{1/2}}\left(\frac{3}{(f_2-f_1)^2} - \frac{8}{A}\varepsilon x^1\right), \quad \underset{4}{\xi}^3 = \frac{1}{2(f_2-f_1)^{1/2}}\left(\frac{1}{f_2-f_1} - \frac{4}{A}\varepsilon x^2\right),$$

$$\underset{4}{\xi}^4 = \frac{1}{A(f_2-f_1)^{1/2}}, \quad \underset{5}{\xi}^5 = \frac{1}{(f_1-f_2)^2}, \quad (A = 3\varepsilon\left(x^3+\omega(x^4)\right) + 1 - \varepsilon).$$

The canonical forms a_{ij} and g_{ij} are given by the formulas

$$\bar{a}_{pq} = \begin{pmatrix} 0 & 0 & 0 & e_1 f_1 & 0 \\ 0 & 0 & e_1 f_1 & e_1 & 0 \\ 0 & e_1 f_1 & e_1 & 0 & 0 \\ e_1 f_1 & e_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_2 f_2 \end{pmatrix}, \quad \bar{g}_{pq} = \begin{pmatrix} 0 & 0 & 0 & e_1 & 0 \\ 0 & 0 & e_1 & 0 & 0 \\ 0 & e_1 & 0 & 0 & 0 \\ e_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & e_2 \end{pmatrix}, \tag{6}$$

and the following equations are satisfied:

$$Y_1\varphi = Y_2\varphi = Y_3\varphi = 0, \tag{7}$$

$$df_1 = \frac{1}{2}(Y_4\varphi)\theta_1, \quad df_2 = 2(Y_5\varphi)\theta_5, \tag{8}$$

$$\omega_{13} = \frac{1}{2}(Y_4\varphi)\theta_1, \quad \omega_{14} = -(Y_4\varphi)\theta_2, \quad \omega_{15} = \frac{Y_5\varphi}{f_2-f_1}\theta_1,$$

$$\omega_{24} = -(Y_4\varphi)\theta_3, \quad \omega_{25} = \frac{Y_5\varphi}{(f_2-f_1)^2}\theta_1 + \frac{Y_5\varphi}{f_2-f_1}\theta_2, \tag{9}$$

$$\omega_{34} = -(Y_4\varphi)\theta_4, \quad \omega_{35} = \frac{Y_5\varphi}{(f_2-f_1)^3}\theta_1 + \frac{Y_5\varphi}{(f_2-f_1)^2}\theta_2 + \frac{Y_5\varphi}{f_2-f_1}\theta_3,$$

$$\omega_{45} = \frac{Y_5\varphi}{(f_2-f_1)^4}\theta_1 + \frac{Y_5\varphi}{(f_2-f_1)^3}\theta_2 + \frac{Y_5\varphi}{(f_2-f_1)^2}\theta_3 + \frac{Y_5\varphi}{f_2-f_1}\theta_4 + \frac{Y_4\varphi}{f_2-f_1}\theta_5.$$

Here

$$\varphi = 2f_1 + \frac{1}{2}f_2 \tag{10}$$

is the defining function of a projective motion of type $\{41\}$, $\omega_{ij} = \gamma_{jik}\theta^k$ is the 1-form of connection in the skew-normal frame (Y_h), $f_1 = \varepsilon x^4 + (1-\varepsilon)\kappa$, $f_2 = f_2(x^5)$, ε takes values 0 and 1, κ is a constant, $e_1, e_2 = \pm 1$. [4]. Using the first Cartan structure equation $d\theta_i = -\sum_{j=1}^{5} e_j\omega_{ij}\wedge\theta_{\bar{j}}$ and the formulas (9), we find

$$d\theta_1 = -\frac{3e_1(Y_4\varphi)}{2}\theta_1\wedge\theta_2 - \frac{e_2(Y_5\varphi)}{f_2-f_1}\theta_1\wedge\theta_5,$$

$$d\theta_2 = -e_1(Y_4\varphi)\theta_1\wedge\theta_3 - \frac{e_2(Y_5\varphi)}{(f_2-f_1)^2}\theta_1\wedge\theta_5 - \frac{e_2(Y_5\varphi)}{f_2-f_1}\theta_2\wedge\theta_5,$$

$$d\theta_3 = -\frac{e_1(Y_4\varphi)}{2}\theta_1\wedge\theta_4 - \frac{e_2(Y_5\varphi)}{(f_2-f_1)^3}\theta_1\wedge\theta_5 - \frac{e_2(Y_5\varphi)}{(f_2-f_1)^2}\theta_2\wedge\theta_5 - \frac{e_2(Y_5\varphi)}{f_2-f_1}\theta_3\wedge\theta_5,$$

$$d\theta_4 = -\frac{e_2(Y_5\varphi)}{(f_2-f_1)^4}\theta_1\wedge\theta_5 - \frac{e_2(Y_5\varphi)}{(f_2-f_1)^3}\theta_2\wedge\theta_5 - \frac{e_2(Y_5\varphi)}{(f_2-f_1)^2}\theta_3\wedge\theta_5 - \frac{e_2(Y_5\varphi)}{f_2-f_1}\theta_4\wedge\theta_5,$$

$$d\theta_5 = -\frac{e_1(Y_4\varphi)}{f_2-f_1}\theta_1\wedge\theta_5.$$

We introduce the notation

$$A_1 \equiv Y_4\varphi, \quad A_2 \equiv Y_5\varphi,$$

$$C_1 \equiv e_1 Y_4(Y_4\varphi), \quad C_2 \equiv e_2 Y_5(Y_5\varphi). \tag{11}$$

$$d\omega_{12} = 0, \; d\omega_{13} = 0, \quad d\omega_{14} = -C_1\theta_1\wedge\theta_2 + A_1^2\theta_1\wedge\theta_3 + \frac{e_2 A_1 A_2}{(f_2-f_1)^2}\theta_1\wedge\theta_5,$$

$$d\omega_{15} = -\frac{3e_1A_1A_2}{2(f_2-f_1)}\theta_1\wedge\theta_2 - \left(\frac{C_2}{f_2-f_1} - \frac{e_2A_2^2}{(f_2-f_1)^2}\right)\theta_1\wedge\theta_5, \quad d\omega_{23}=0,$$

$$d\omega_{24} = -C_1\theta_1\wedge\theta_3 + \frac{e_1A_1^2}{2}\theta_1\wedge\theta_4 + \frac{e_2A_1A_2}{(f_2-f_1)^3}\theta_1\wedge\theta_5 + \frac{3e_1A_1^2}{2}\theta_2\wedge\theta_3 + \frac{e_2A_1A_2}{(f_2-f_1)^2}\theta_2\wedge\theta_5,$$

$$d\omega_{25} = -\frac{e_1A_1A_2}{f_2-f_1}\theta_1\wedge\theta_3 - \left(\frac{C_2}{(f_2-f_1)^2} - \frac{2e_2A_2^2}{(f_2-f_1)^3}\right)\theta_1\wedge\theta_5 - \left(\frac{C_2}{f_2-f_1} - \frac{e_2A_2^2}{(f_2-f_1)^2}\right)\theta_2\wedge\theta_5,$$

$$d\omega_{34} = -C_1\theta_1\wedge\theta_4 + \frac{e_2A_1A_2}{(f_2-f_1)^4}\theta_1\wedge\theta_5 + \frac{3e_1A_1^2}{2}\theta_2\wedge\theta_4 + \frac{e_2A_1A_2}{(f_2-f_1)^3}\theta_2\wedge\theta_5 + \frac{e_2A_1A_2}{(f_2-f_1)^2}\theta_3\wedge\theta_5,$$

$$d\omega_{35} = \frac{e_1A_1A_2}{2(f_2-f_1)^3}\theta_1\wedge\theta_2 + \frac{e_1A_1A_2}{2(f_2-f_1)^2}\theta_1\wedge\theta_3 - \frac{e_1A_1A_2}{2(f_2-f_1)}\theta_1\wedge\theta_4 - \left(\frac{C_2}{(f_2-f_1)^3} - \frac{3e_2A_2^2}{(f_2-f_1)^4}\right)\theta_1\wedge\theta_5 - \left(\frac{C_2}{(f_2-f_1)^2} - \frac{2e_2A_2^2}{(f_2-f_1)^3}\right)\theta_2\wedge\theta_5 - \left(\frac{C_2}{f_2-f_1} - \frac{e_2A_2^2}{(f_2-f_1)^2}\right)\theta_3\wedge\theta_5,$$

$$d\omega_{45} = \frac{e_1A_2A_1}{(f_2-f_1)^4}\theta_1\wedge\theta_2 + \frac{e_1A_2A_1}{(f_2-f_1)^3}\theta_1\wedge\theta_3 + \frac{e_1A_2A_1}{(f_2-f_1)^2}\theta_1\wedge\theta_4 + \left(\frac{C_1}{(f_2-f_1)} - \frac{C_2}{(f_2-f_1)^4} + \frac{4e_2A_2^2}{(f_2-f_1)^5} - \frac{e_1A_1^2}{2(f_2-f_1)^2}\right)\theta_1\wedge\theta_5 - \left(\frac{C_2}{(f_2-f_1)^3} + \frac{3e_1A_1^2}{2(f_2-f_1)} - \frac{3e_2A_2^2}{(f_2-f_1)^4}\right)\theta_2\wedge\theta_5 - \left(\frac{C_2}{(f_2-f_1)^2} - \frac{2e_2A_2^2}{(f_2-f_1)^3}\right)\theta_3\wedge\theta_5 - \left(\frac{C_2}{f_2-f_1} - \frac{e_2A_2^2}{(f_2-f_1)^2}\right)\theta_4\wedge\theta_5$$

By taking the external differential from both sides of the equation $df_1 = e_1(Y_4\varphi)\theta_1 \equiv e_1A_1\theta_1$ (8) and comparing this to $dA_1 = \theta^l Y_l Y_4\varphi = \theta^l[Y_l, Y_4] + \theta^l Y_4 Y_l\varphi$, we get by using (7)

$$dA_1 = C_1\theta_1 - \frac{3}{2}e_1A_1^2\theta_2 - \frac{e_2A_1A_2}{f_2-f_1}\theta_5;$$

dA_2 is similarly calculated.

Using the obtained relations and the second Cartan structure equation $\Omega_{ij} = d\omega_{ij} + \sum_{l=1}^{5} e_l\omega_{il}\wedge\omega_{\tilde{l}j}$, we calculate the curvature 2-form Ω_{ij} of h-space of type $\{41\}$:

$$\Omega_{12} = -\frac{e_2A_2^2}{(f_2-f_1)^2}\theta_1\wedge\theta_2, \quad \Omega_{13} = \left(\frac{e_1A_1^2}{2} - \frac{e_2A_2^2}{(f_2-f_1)^3}\right)\theta_1\wedge\theta_2 - \frac{e_2A_2^2}{(f_2-f_1)^2}\theta_1\wedge\theta_3,$$

$$\Omega_{14} = -\left(C_1 + \frac{e_2A_2^2}{(f_2-f_1)^4}\right)\theta_1\wedge\theta_2 + \left(\frac{e_1A_1^2}{2} - \frac{e_2A_2^2}{(f_2-f_1)^3}\right)\theta_1\wedge\theta_3 - \frac{e_2A_2^2}{(f_2-f_1)^2}\theta_1\wedge\theta_4,$$

$$\Omega_{15} = -\left(\frac{C_2}{f_2-f_1} - \frac{e_2A_2^2}{(f_2-f_1)^2}\right)\theta_1\wedge\theta_5, \quad \Omega_{23} = \left(\frac{e_1A_1^2}{2} - \frac{e_2A_2^2}{(f_2-f_1)^3}\right)\theta_1\wedge\theta_3 - \frac{e_2A_2^2}{(f_2-f_1)^2}\theta_2\wedge\theta_3, \quad \Omega_{24} = -\left(C_1 + \frac{e_2A_2^2}{(f_2-f_1)^4}\right)\theta_1\wedge\theta_3 + \left(\frac{e_1A_1^2}{2} - \frac{e_2A_2^2}{(f_2-f_1)^3}\right)\theta_1\wedge\theta_4 +$$

$$\left(\frac{e_1A_1^2}{2}-\frac{e_2A_2^2}{(f_2-f_1)^3}\right)\theta_2\wedge\theta_3-\frac{e_2A_2^2}{(f_2-f_1)^2}\theta_2\wedge\theta_4, \tag{12}$$

$$\Omega_{25}=-\left(\frac{C_2}{(f_2-f_1)^2}-\frac{2e_2A_2^2}{(f_2-f_1)^3}\right)\theta_1\wedge\theta_5-\left(\frac{C_2}{f_2-f_1}-\frac{e_2A_2^2}{(f_2-f_1)^2}\right)\theta_2\wedge\theta_5,$$

$$\Omega_{34}=-\left(C_1+\frac{e_2A_2^2}{(f_2-f_1)^4}\right)\theta_1\wedge\theta_4+\left(\frac{e_1A_1^2}{2}-\frac{e_2A_2^2}{(f_2-f_1)^3}\right)\theta_2\wedge\theta_4-\frac{e_2A_2^2}{(f_2-f_1)^2}\theta_3\wedge\theta_4,$$

$$\Omega_{35}=-\left(\frac{C_2}{(f_2-f_1)^3}+\frac{e_1A_1^2}{2(f_2-f_1)}-\frac{3e_2A_2^2}{(f_2-f_1)^4}\right)\theta_1\wedge\theta_5-$$
$$\left(\frac{C_2}{(f_2-f_1)^2}-\frac{2e_2A_2^2}{(f_2-f_1)^3}\right)\theta_2\wedge\theta_5-\left(\frac{C_2}{f_2-f_1}-\frac{e_2A_2^2}{(f_2-f_1)^2}\right)\theta_3\wedge\theta_5,$$

$$\Omega_{45}=\left(\frac{C_1}{(f_2-f_1)}-\frac{C_2}{(f_2-f_1)^4}+\frac{4e_2A_2^2}{(f_2-f_1)^5}-\frac{e_1A_1^2}{2(f_2-f_1)^2}\right)\theta_1\wedge\theta_5-$$
$$\left(\frac{C_2}{(f_2-f_1)^3}+\frac{e_1A_1^2}{2(f_2-f_1)}-\frac{3e_2A_2^2}{(f_2-f_1)^4}\right)\theta_2\wedge\theta_5-$$
$$\left(\frac{C_2}{(f_2-f_1)^2}-\frac{2e_2A_2^2}{(f_2-f_1)^3}\right)\theta_3\wedge\theta_5-\left(\frac{C_2}{f_2-f_1}-\frac{e_2A_2^2}{(f_2-f_1)^2}\right)\theta_4\wedge\theta_5.$$

2. H-spaces (H_{41}, g) of constant curvature.

Theorem 1. *The necessary and sufficient condition for an h-space (H_{41}, g) of type $\{41\}$ to be a space of constant curvature K: $\Omega_{ij}=K\theta_i\wedge\theta_j$, is $K_{1312}=0$, which is equivalent to*

$$D\equiv\frac{e_1A_1^2}{2}-\frac{e_2A_2^2}{(f_2-f_1)^3}=0, \tag{13}$$

moreover, $K_{ijkl}=0$, $\rho_{ij}=0$ for all (ij) and $(kl)\neq(ij)$, that is $\Omega_{ij}=0$, and any h-space H_{41} of type $\{41\}$ of constant curvature is flat.

P r o o f. We write the curvature 2-form as

$$\Omega_{ij}\equiv\sum_{(kl)}K_{ijkl}\theta_k\wedge\theta_l \quad (k,l=1,...,5,\ k<l) \tag{14}$$

and put $K_{ijij}\equiv\rho_{ij}$, then the formula (14) takes the form

$$\Omega_{ij}=\rho_{ij}\theta_i\wedge\theta_j+\sum_{(kl)\neq(ij)}K_{ijkl}\theta_k\wedge\theta_l \quad (i,j,k,l=1,...,5,\ i<j,\ k<l),$$

where, by virtue of (12),

$$\rho_{12}=-\frac{e_2A_2^2}{(f_2-f_1)^2}=\rho_{13}=\rho_{14}=\rho_{23}=\rho_{24}=\rho_{34}, \tag{15}$$

$$\rho_{15}=-\frac{C_2}{f_2-f_1}+\frac{e_2A_2^2}{(f_2-f_1)^2}=\rho_{25}=\rho_{35}=\rho_{45},$$

and among the coefficients K_{ijkl} for $(kl)\neq(ij)$ are nonzero only

$$K_{1312}=\frac{e_1A_1^2}{2}-\frac{e_2A_2^2}{(f_2-f_1)^3},\ K_{1412}=-C_1-\frac{e_2A_2^2}{(f_2-f_1)^4},$$

$$K_{1413} = K_{1312} = K_{2313},\ K_{2413} = K_{1412},\quad K_{2414} = K_{1312} = K_{2423},$$
$$K_{2515} = -\frac{C_2}{(f_2 - f_1)^2} + \frac{2e_2A_2^2}{(f_2 - f_1)^3},\quad K_{3414} = K_{1412},$$
$$K_{3515} = -\frac{C_2}{(f_2 - f_1)^3} - \frac{e_1A_1^2}{2(f_2 - f_1)} + \frac{3e_2A_2^2}{(f_2 - f_1)^4}, \tag{16}$$
$$K_{3525} = K_{2515},\quad K_{4525} = K_{3515},\quad K_{4535} = K_{2515},$$
$$K_{4515} = \frac{C_1}{f_2 - f_1} - \frac{C_2}{(f_2 - f_1)^4} - \frac{e_1A_1^2}{2(f_2 - f_1)^2} + \frac{4e_2A_2^2}{(f_2 - f_1)^5},$$

Equating K_{ijkl} with $(k, l) \neq (i, j)$, $i, j, k, l = 1, ..., 5$, to zero, we get the following five conditions:

$$\frac{e_1A_1^2}{2} - \frac{e_2A_2^2}{(f_2 - f_1)^3} \equiv D = 0, \tag{17}$$
$$-C_1 - \frac{e_2A_2^2}{(f_2 - f_1)^4} = 0, \tag{18}$$
$$-\frac{C_2}{(f_2 - f_1)^2} + \frac{2e_2A_2^2}{(f_2 - f_1)^3} = 0, \tag{19}$$
$$-\frac{C_2}{(f_2 - f_1)^3} - \frac{e_1A_1^2}{2(f_2 - f_1)} + \frac{3e_2A_2^2}{(f_2 - f_1)^4} = 0, \tag{20}$$
$$\frac{C_1}{f_2 - f_1} - \frac{C_2}{(f_2 - f_1)^4} - \frac{e_1A_1^2}{2(f_2 - f_1)^2} + \frac{4e_2A_2^2}{(f_2 - f_1)^5} = 0. \tag{21}$$

Given the formulas (10), (11) and (5), the equation (17) becomes

$$\frac{4e_1\varepsilon^2}{(f_2 - f_1)A^2} - \frac{e_2(f_2')^2}{2(f_2 - f_1)^7} = 0$$

where $f_2' = df_2/dx^5$. Differentiating it with respect to x^3, we find $\varepsilon = 0$, after that from (13) we get $f_2' = 0$, it follows $A_1 = A_2 = 0$, $C_1 \equiv e_1Y_4(A_1) = 0$, $C_2 \equiv e_2Y_5(A_2) = 0$. In this case, the conditions (17)-(21) are carried out identically, moreover, $\rho_{ij} \equiv 0$ for all $i, j = 1, ...5$. Hence, $\Omega_{ij} = 0$ for all $i, j = 1, ...5$, therefore $K = 0$, and h-space (H_{41}, g) is flat. Q.E.D.

3. Integrability conditions of the Eisenhart equation.

Theorem 2. *Any solution (k, g, ψ) of the Eisenhart equation*

$$\nabla k(Y, Z, W) = 2g(Y, Z)W\psi + g(W, Z)Y\psi + g(Y, W)Z\psi,$$

which is equivalent after change $k = b + 2\psi g$ to the equation

$$\nabla b(Y, Z, W) = g(W, Z)Y\psi + g(Y, W)Z\psi, \tag{22}$$

in h-space (H_{41}, g) of type $\{41\}$ of non-constant curvature satisfies the condition

$$\psi = c_1\left(2f_1 + \frac{1}{2}f_2\right) + const = c_1\varphi + const, \tag{23}$$

where the function φ is defined by the relation (10), and c_1 is an arbitrary constant.

P r o o f. In view of the invariance of the quantities f_i and the tensor nature of the equality (23), it suffices to prove it in the canonical skew-frame (5), where the equation (22) takes the form

$$d\bar{b}_{pq} + \sum_{h=1}^{5} e_h\left(\bar{b}_{hq}\omega_{p\tilde{h}} + \bar{b}_{ph}\omega_{q\tilde{h}}\right) = (Y_q\psi)\theta_p + (Y_p\psi)\theta_q, \tag{24}$$

here $\omega_{p\tilde{h}}$ is defined by the formulas (9), and $\bar{b}_{pq}$ are the components of the tensor b in the skew-frame (5).

By differentiating the Eisenhart equation (24), we obtain its integrability conditions:

$$\bar{b}_{ph}\Omega^h{}_q + \bar{b}_{hq}\Omega^h{}_p = \psi_{ph}\theta^h \wedge \theta_q + \psi_{hq}\theta^h \wedge \theta_p, \tag{25}$$

where $\Omega^h{}_p = e_h\Omega_{\tilde{h}p}$, $\psi_{ph} \equiv -Y_hY_p\psi - \gamma^l{}_{ph}Y_l\psi = \psi_{hp}$. By equating the coefficients for identical basic 2-forms $\theta_\alpha \wedge \theta_\beta$ left and right in (25), for $(pq) = (13)$ and $(\alpha\beta) = (24)$ we find

$$\left(\frac{e_1A_1^2}{2} - \frac{e_2A_2^2}{(f_2 - f_1)^3}\right)\bar{b}_{11} = 0.$$

If $\bar{b}_{11} \neq 0$, then it follows (13), and, by Theorem 1, (H_{41}, g) is a space of constant curvature. As this contradicts the assumption, we have $\bar{b}_{11} = 0$. We similarly get

$$\bar{b}_{12} = \bar{b}_{13} = \bar{b}_{15} = \bar{b}_{22} = \bar{b}_{25} = \bar{b}_{34} = \bar{b}_{35} = \bar{b}_{44} = \bar{b}_{45} = 0. \tag{26}$$

From the equation (24), where ω_{hs} are defined by the formulas (9), (10) and (5), for $(pq) = (11), (12), (33), (34)$ and (35), using the equalities (26) and considering that $\underset{4}{\xi}^4 \neq 0$, $\underset{5}{\xi}^5 \neq 0$, we find

$$\underset{1}{\xi}^1\partial_1\psi = 0,\ \underset{2}{\xi}^1\partial_1\psi + \underset{2}{\xi}^2\partial_2\psi = 0,\ \underset{3}{\xi}^1\partial_1\psi + \underset{3}{\xi}^2\partial_2\psi + \underset{3}{\xi}^3\partial_3\psi = 0, \tag{27}$$

$$Y_i\bar{b}_{33} = 0\ (i = 1, ..., 5), \tag{28}$$

$$\partial_4\psi = 2e_1\bar{b}_{33}f_1', \tag{29}$$

$$\partial_5\psi = \frac{1}{2}e_1\bar{b}_{33}f_2'. \tag{30}$$

From the equation (27) we derive $\psi = \psi(x^4, x^5)$ by using the formulas (5). Then from the equation (28) it follows $\bar{b}_{33} = e_1c_1 = const$. By integrating the equations (29) and (30), we obtain

$$\psi = c_1\left(2f_1 + \frac{f_2}{2}\right) + \mathrm{const} = c_1\varphi + const.$$

Q.E.D.

4. Main theorems.

Theorem 3. *Any covariantly constant symmetric tensor b_{ij} in h-space (H_{41}, g) of type $\{41\}$ of non-constant curvature is proportional to the metric tensor:*

$$b_{ij} = c_2g_{ij} \qquad (c_2 = const).$$

P r o o f. In the skew-normal frame (5) the equation $b_{ij,k} = 0$ takes the form

$$d\bar{b}_{pq} + \sum_{h=1}^{5} e_h\left(\bar{b}_{hq}\omega_{p\tilde{h}} + \bar{b}_{ph}\omega_{q\tilde{h}}\right) = 0. \tag{31}$$

The integrability conditions for the equations (31) are obtained from (25) for $\psi = const$ and have the form

$$\bar{b}_{ph}\Omega^h{}_q + \bar{b}_{hq}\Omega^h{}_p = 0. \tag{32}$$

Hence, as in the previous case, we obtain the equalities (26) in h-space (H_{41}, g) of non-constant curvature. From (32) for $(pq) = (14), (33)$ it follows $D\bar{b}_{24} = D\bar{b}_{33} = 0$, and since in h-space (H_{41}, g) of non-constant curvature $D \neq 0$, then $\bar{b}_{24} = \bar{b}_{33} = 0$. From (31) for $(pq) = (14), (23), (55)$ we find $d\bar{b}_{14} = 0$, $d\bar{b}_{23} = 0$ and $d\bar{b}_{55} = 0$, whence it follows that $\bar{b}_{14}$, $\bar{b}_{23}$ and $\bar{b}_{55}$ are constant.

From (32) for $(pq) = (13)$ we find

$$D(\bar{b}_{14} - \bar{b}_{23}) = 0,$$

from here for $D \neq 0$ we have $\bar{b}_{14} = \bar{b}_{23}$, after that from (31) for $(pq) = (45)$ we derive

$$(e_2\bar{b}_{55} - e_1\bar{b}_{14})A_2 = 0,$$

$$(e_2\bar{b}_{55} - e_1\bar{b}_{14})A_1 = 0.$$

If $(e_2\bar{b}_{55} - e_1\bar{b}_{14}) \neq 0$ then $A_1 = A_2 = 0$; this implies (13) and by Theorem 1 (H_{41}, g) has constant curvature, which contradicts the assumption. Therefore, $e_1\bar{b}_{14} = e_1\bar{b}_{23} = e_2\bar{b}_{55}$. Putting $\bar{b}_{14} = e_1c_2$ we find finally $\bar{b}_{pq} = c_2\bar{g}_{pq}$, where $(\bar{g}_{pq})$ is defined by (6). Q.E.D.

Since the vector field X is an affine motion in (H_{41}, g), if and only if $(L_Xg)_{,k} = 0$, then the theorem (3) implies

Theorem 4. *Every affine motion X in an h-space (H_{41}, g) of type $\{41\}$ of non-constant curvature is an infinitesimal homothety: $L_Xg = cg,\ c = const.$*

Since any two solutions h_1 and h_2 of the Eisenhart equation (3) with the same right-hand side can differ only by the covariantly constant tensor b, from the theorem 2 and the linearity of the equation (3) it follows that the general solution of the Eisenhart equation in an h-space (H_{41}, g) of non-constant curvature can be written in the form $c_1h + b$ or, by virtue of the theorem 3, in the form $c_1h + c_2g$, where $h = a + 2\varphi g$, g and a are defined in the skew-normal frame (5) by canonical forms (6) [4], c_1, c_2 are constant. From here it follows

Theorem 5. *A vector field X is a projective motion in an h-space (H_{41}, g) of non-constant curvature if and only if*

$$L_Xg = c_1h + c_2g \equiv c_1(a + 2\varphi g) + c_2g,$$

where φ is the defining function of the projective motion X, g and a are defined in the skew-normal frame (5) by canonical forms (6), c_1, c_2 are arbitrary constants.

Theorem 5 implies

Theorem 6. *If an h-space (H_{41}, g) of type $\{41\}$ of non-constant curvature admits a r-dimensional non-homothetic projective Lie algebra P_r, then this algebra contains a $(r-1)$-dimensional homothetic subalgebra.*

P r o o f. If $(X_1, \ldots, X_r)$ is the basis of the Lie algebra P_r, then $L_{X_s}g = \underset{s}{c}_1h + \underset{s}{c}_2g$, $s = 1, \ldots, r$, where one of the constants $\underset{s}{c}_1$, for example, $\underset{1}{c}_1$ is nonzero (otherwise P_r consists of homotheties). In the new basis $Z_1 = X_1$, $Z_\tau = \underset{1}{c}_1X_\tau - \underset{\tau}{c}_1X_1$ we have $L_{Z_\tau}g = (\underset{1}{c}_1\underset{\tau}{c}_2 - \underset{1}{c}_2\underset{\tau}{c}_1)g$, $\tau = 2, \ldots, r$. Q.E.D.

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